Prob. of finding e at any position must be 1 ⇒ Normalized Cond.:

$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} \psi(x)^* \psi(x)\, dx = 1$

Normalized Cond. allows to find amplitude

$k = \frac{2\pi}{\lambda}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\psi(x) = C\cos(kx) + D\sin(kx)$

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ $\quad p = \hbar k$

Schrödinger Eq. of motion describes how wave function depends on space & time:

$\frac{-\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$ (total E)

$(K(x) + U(x))\psi(x) = E\psi(x)$

when $U(x) = 0$, $E_{total} = K.E$

∴ sol'n to S.E is → $K(Ae^{ikx} + Be^{-ikx}) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}(Ae^{ikx} + Be^{-ikx}) = \frac{\hbar^2 k^2}{2m}(Ae^{ikx} + Be^{-ikx})$

It is thus $\psi(x) = Ae^{ikx} + Be^{-ikx}$

Momentum: set $B=0$, $\lambda = h/p$, $k = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = p\frac{2\pi}{h} = \frac{p}{\hbar}$; $\hbar =$

$\hat{p}\psi(x) = -i\hbar\frac{d}{dx}\psi(x) = -i\hbar\frac{d}{dx}Ae^{ikx} = -i\hbar A\frac{d}{dx}e^{ikx} = -i\hbar A(i$

$= \hbar k Ae^{ikx} = \hbar k\psi(x) = p\psi(x)$; $\psi(x) = Be^{-ikx}$ describes freely moving particle moving in −x-component w/ −p. Superposition of left & right moving waves

Kinetic Energy: $\hat{K}\psi(x) = \frac{1}{2m}\hat{p}^2\psi(x) = \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2\psi(x)$

$= \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}Ae^{ikx} = -ik\frac{\hbar^2}{2m}\frac{d}{dx}Ae^{ikx} = k^2\frac{\hbar^2}{2m}$

$= \frac{p^2}{2m}Ae^{ikx} = KAe^{ikx}$; $\psi(x) = Be^{-ikx}$ gives same value

Sol'n to SE for P.E >> 0 use the limit of infinite P.E.

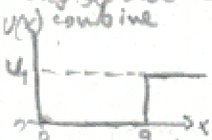
For ∞ P.E @ some pt in space, SE requires $E = \infty$ or $\psi(x)$ has value zero ∴ $\psi(x) = 0$ in that region. Impossible to be in region w/ ∞ P.E — "Forbidden Regions"

→ wave funct = continuous — wave func = 0 region w/ ∞ P.E — wave funct must be normalized

Finite Potential Well

*Find sol'n for all 3 reg. separate then combine

$U_x = \begin{cases} \infty & \text{for } x<0 \\ 0 & \text{for } 0 \le x \le a \\ U_1 & \text{for } x>a \end{cases}$



For $x<0$, wave funct must have constant value $= 0$

For 0 to a, wave funct. has general form $\psi(x) = C\cos(kx) + D\sin(kx)$, again sol'n (C_n)

For $x>a$, SE in this region: $\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U_1\psi(x)$

Rearranging: $\frac{d^2\psi(x)}{dx^2} = \frac{2m(U_1 - E)}{\hbar^2}\psi(x)$ $= E\psi(x)$

*Case 1: Energy larger than well depth $E > U_1$, $\frac{2m(U_1 - E)}{\hbar^2} < 0$ we get oscillatory sol'n

$\psi(x) = E\cos(k'x) + F\sin(k'x)$ for $x>a$ & $E>U_1$

Same kind as b/w 0 & a but w/ diff λ + ∴ diff wave # k

inside interval [0, a], no P.E ∴ all K.E, $E = \frac{\hbar^2 k^2}{2m}$, K.E ⇒ $E - U_1 = \frac{\hbar^2 k'^2}{2m}$

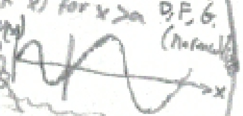
∴ $k' = \sqrt{k^2 - \frac{2mU_1}{\hbar^2}} = \sqrt{\frac{2m(E-U_1)}{\hbar^2}}$

wave function for $E_{tot} > U_1$ is:

$\psi(x) = \begin{cases} 0 & \text{for } x<0 \\ D\sin(kx) & \text{for } 0 \le x \le a \\ F\cos(k'x) + G\sin(k'x) & \text{for } x>a \end{cases}$

Now find values of amplitudes D, F, G. (Normalize)

*Case 2: Energy smaller than the well, Bound States



For $E < U_1$, $\frac{2m(U_1 - E)}{\hbar^2} > 0$, so instead of oscill sol'n, we get exponential sol'n to differential eq.

$\frac{d^2\psi(x)}{dx^2} = \frac{2m(U_1 - E)}{\hbar^2}\psi(x)$; $\gamma = \sqrt{\frac{2m(U_1 - E)}{\hbar^2}}$ ⇒ $\frac{d^2\psi(x)}{dx^2} = \gamma^2\psi(x)$

Has sol'n: $\psi(x) = Fe^{-\gamma x} + Ge^{\gamma x}$ for $x>a$, $E<U_1$

discard rising expo $e^{\gamma x}$ since it becomes ∞ as $x \to \infty$ (otherwise it wouldn't normalize) ∴ $G = 0$, sol'n becomes

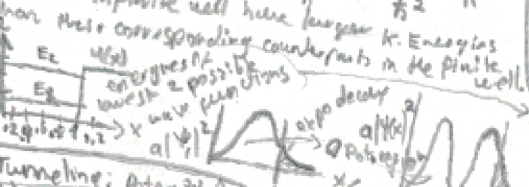
$\psi(x) = Fe^{-\gamma x}$ for $x>a$, $E<U_1$ wave funct for all 3 regions: when P.E = 0 in reg. $0 \le x \le a$, $E = \frac{\hbar^2 k^2}{2m}$

$\psi(x) = \begin{cases} 0 & \text{for } x<0 \\ D\sin(kx) & \text{for } 0 \le x \le a \\ Fe^{-\gamma x} & \text{for } x>a \end{cases}$

Conditions to Normalize: 1) continuous 2) has continuous deriv @ x=0

$\gamma^2 = \frac{2m(U_1 - E)}{\hbar^2} = \frac{2mU_1}{\hbar^2} - \frac{2mE}{\hbar^2} = \frac{2mU_1}{\hbar^2} - k^2$

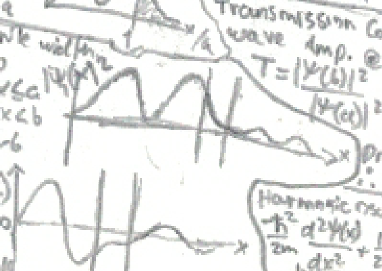
sol'n in infinite well have larger K. Energies than their corresponding counterparts in the finite well.



Tunneling: Potential step has finite width

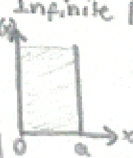
$U(x) = \begin{cases} \infty & \text{for } x<0 \\ 0 & \text{for } 0 \le x \le a \\ U_1 & \text{for } a<x<b \\ 0 & \text{for } x>b \end{cases}$

$\psi(x) = \begin{cases} 0 & \text{for } x<0 \\ D\sin(kx) & \text{for } 0 \le x \le a \\ Fe^{-\gamma x} & \text{for } a<x<b \\ G\sin(kx+\phi) & \text{for } x>b \end{cases}$



Beyond $x=b$, it oscillates again, this is "tunneling": classically, particle @ $0<x<a$ w/ some E_{total} would not have enough E to escape & would be trapped in reg. $0 \le x \le a$

Infinite Potential Well



$U(x) = \begin{cases} \infty & \text{for } x<0 \\ 0 & \text{for } 0 \le x \le a \\ \infty & \text{for } x>a \end{cases}$

$\psi(0) = C\cos(k\cdot 0) + D\sin(k\cdot 0) = C = 0$; $\psi(x) = D\sin(kx)$ gives a sol'n b/w 0 & a. Left-travelling Be^{-ikx} & Right travelling wave Ae^{ikx} results in standing wave

⇒ $\psi(x) = \begin{cases} 0 & \text{for } x<0 \\ D\sin(kx) & \text{for } 0 \le x \le a \\ 0 & \text{for } x>a \end{cases}$

boundary condition: $\psi(x=a) = D\sin(ka)$

implies not all values of $\lambda = 2\pi/k$ are possible, since integer × π will always give sin funct = 0. Only $k = \frac{2\pi}{\lambda}$; $ka = \frac{2\pi a}{\lambda} = n\pi$, $n = 1, \dots$

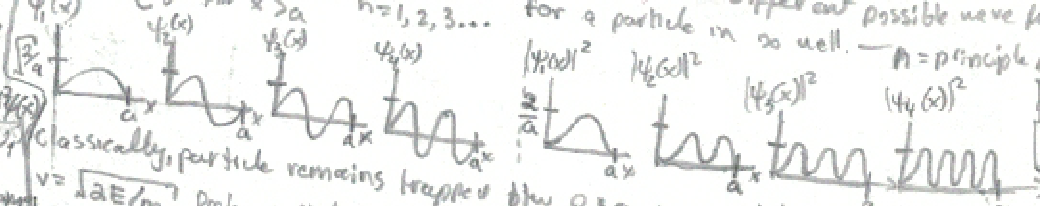
Now find amplitude D → normalize to do this: ∴ Sol'n $\psi(x) = \begin{cases} 0 \\ D\sin\left(\frac{n\pi x}{a}\right) \\ 0 \end{cases}$

$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{0} |\psi(x)|^2 dx + \int_{0}^{a} |\psi(x)|^2 dx + \int_{a}^{\infty} |\psi(x)|^2 dx = 0 + \int_0^a \left|D\sin\left(\frac{n\pi x}{a}\right)\right|^2 dx + \dots$

$= |D|^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = |D|^2\frac{a}{2}$ ∴ $|D|^2 = \frac{2}{a}$ ⇒ $D = \sqrt{\frac{2}{a}}$ ⇒ $D = \sqrt{\frac{2}{a}}e^{i\theta}$

$\psi(x) = \begin{cases} 0 & \text{for } x<0 \\ \sqrt{2/a}\sin\left(\frac{n\pi x}{a}\right) & \text{for } 0 \le x \le a \\ 0 & \text{for } x>a \end{cases}$ $\quad n = 1, 2, 3\dots$

n corresponds to different possible wave functions for a particle in ∞ well. — n = principle Q.#

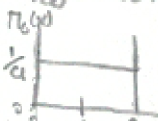


Classically, particle remains trapped b/w 0 & a & moves between them w/ constant $v = \sqrt{2E/m}$. Prob of find particle b/w x & x+dx is $\Pi_c(x)dx \propto \frac{1}{v(x)}$ — ind. of position

Energy of Particle in Infinite Potential well: Inside interval 0 & a, no P.E ∴ $E_{tot} = k.E$.

$k^2\psi(x) = \frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}$ For each $k \neq n$, a different result should be for KE & ∴ total E:

avg of this oscillation, while the prob distribution oscillates b/w 0 & max value 2/a



$E_n\psi_n(x) = \frac{-\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\left(\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\right) = \frac{-\hbar^2}{2m}\left(\frac{n\pi}{a}\right)\frac{d}{dx}\left(\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right)\right)$

$= \frac{-\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2\left(\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\right) = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2\psi_n(x)$ ∴ $E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}$ $\quad n = 1, 2, 3\dots$

4 lowest levels ∞ Potential Well

Classically the e⁻ would bounce at the walls & could be found @ some pt b/w x=0 & x=a, quantum world allows e⁻ to penetrate potential Boundary into classically forbidden region x>a

Transmission Coefficient: ratio of abs. square of wave amp. @ exit of barrier to abs square of wave amp. @ entrance of barrier. For wave funct. $Fe^{-\gamma x}$ in region of barrier, T coefficient:

$T = \frac{|\psi(b)|^2}{|\psi(a)|^2} = \frac{|Fe^{-\gamma b}|^2}{|Fe^{-\gamma a}|^2} = |e^{-\gamma(b-a)}|^2 = e^{-2\gamma(b-a)}$; T. Coef depends on width of barrier b−a

Prob of particle to left of barrier ($x \le a$) ∝ to $|\psi(a)|^2$ & right of barrier ($x \ge b$) ∝ ...

∴ T Coef. measures prob. that a particle hitting barrier on left will emerge on right

Harmonic oscillator: $E_n = (n + \frac{1}{2})\hbar\omega_0$ $\quad n = 0, 1, 2\dots$

$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\psi(x) = E\psi(x)$

$\frac{d^2\psi(x)}{dx^2} = -\left(\frac{2mE}{\hbar^2} - \frac{m^2\omega_0^2}{\hbar^2}x^2\right)\psi(x)$

Ch 4: $\left[\frac{-\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r}\right]\psi = E\psi$ → E of e⁻ for H: $E_n = -\frac{13.6\text{eV}}{n^2}$

$dP = P(r)^2 dr$ prob of particle being in shell of thickness dr

$\langle f(r)\rangle = \int_0^\infty f(r)\,dP = \int_0^\infty f(r)P(r)\,dr$

KE ... particle will not penetrate (classically) in QM, a penetration ψ will not be ...

$\frac{g_i}{g_j} e^{-\Delta E/k_BT}$ MB Distr. $n_i = \frac{N_i}{N} = \frac{g_i e^{-E_i/k_BT}}{Z} = \frac{g_i}{e^{(E_i-\mu)/k_BT}}$ N total # of particles, expected # of particles

prob of finding particle: $f_{MB}(E) = \frac{1}{a e^{E/k_BT}}$

BE Dist. # of particles w/ given energy state E_i: $N_i = \frac{1}{e^{(E_i-\mu)/k_BT}-1}$, $\frac{n_r}{g_r} = \frac{1}{e^{\alpha} e^{E_r/k_BT}-1}$, $f_{BE}(E) = \frac{1}{a e^{E/k_BT}-1}$

FD Dist. $N_i = \frac{g_i}{e^{(E_i-\mu)/k_BT}+1}$, $\frac{n_r}{g_r} = \frac{1}{e^{\alpha} e^{E_r/k_BT}+1}$, $f_{FD}(E) = \frac{1}{a e^{E/k_BT}+1}$

BE $w_r = \frac{(n_r+g_r-1)!}{g_r-1}$; FD $w_r = \binom{g_r}{n_r} = \frac{g_r!}{n_r!(g_r-n_r)!}$ assigning n_r particles

Planck's Radiation Law: $I_T(\lambda) = \frac{2hc^2}{\lambda^5 (e^{hc/\lambda k_BT}-1)}$ ← add π will be $G(\lambda)$

$I_T(f) = \frac{2h}{c^2} \frac{f^3}{e^{hf/k_BT}-1}$ add π will be $E(f)$

$\int U(f)\,df = \frac{8\pi}{c^3} \frac{hf^3}{e^{hf/k_BT}-1}$; let $x = \frac{hf}{k_BT} \to f = \frac{k_BT}{h}x$; $dx = \frac{h\,df}{k_BT} \to df = \frac{k_BT}{h}dx$

$\frac{8\pi h}{c^3} \int \left(\frac{k_BT}{h}\right)^3 \frac{x^3}{e^x - 1} \frac{k_BT}{h} dx$; $\int \frac{x^{n-1}}{e^x-1}dx = \Gamma(n)\,\zeta(n)$, $n=4$, $(n-1)!$

$\left(\frac{8\pi h}{c^3}\right)\left(\frac{k_BT}{h}\right)^4 \int \frac{x^{4-1}}{e^x-1}dx$; $(4-1)!\left(\frac{\pi^4}{90}\right)$

$U(f)\,df = n(f)\,df\,\bar{E}$; $= \frac{N}{Z} \frac{V 8\pi f^2 df}{c^3}$; classically: $\bar{E} = \int E e^{-E/k_BT}$ ... $= \frac{N}{Z} \frac{8\pi k_BT}{c^3} \int_0^\infty f^2\,df = \infty$

$\int e^{-E/k_BT} = k_BT$ (all walls), radiation UV catastrophe

$E_{BJ} = \frac{2\pi c k_BT}{\lambda^4}$, $\lambda \to 0$; Planck: discrete energy; $\ln W$ can't take ... $E = n\epsilon_0$, $\bar{E} = \frac{\sum n\epsilon_0 e^{...}}{\sum e^{...}}$ some math = planck's law

Bose Einstein Condensate: large fraction of atoms go to lowest energy quantum states when atoms are close enough that the deBroglie λ overlap one another. Carl Weinmann & Eric Cornell used magnetic trap to collect rubidium atoms, which ... boson bosons @ low T & in strong magnetic field. ... the part condenses, the rest remains ... ~~gas~~ ... saturated gas

$P(v)\,dv = 4\pi \left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2 e^{-mv^2/2k_BT}\,dv$ Prob. particle has speed btw v & $v+dv$

To find $v_{avg} = \int v P(v)\,dv$; To find v_P, set der. $= 0$; $C_1, C_2 = \frac{1}{2}m/k_BT$; $f(v) = C_1 v^2 e^{-C_2 v^2}$ ← set der. $= 0$

v_{rms}, higher T, more stretched graph

$P(p)\,dp = 4\pi p^2 \left(\frac{1}{2\pi m k_BT}\right)^{3/2} e^{-p^2/2mk_BT}\,dp$ prob. particle has momentum btw p & $p+dp$

$P(E)\,dE = \frac{2}{\sqrt{\pi}(k_BT)^{3/2}} E^{1/2} e^{-E/k_BT}$ prob. particle w/ energy E & $E+dE$

$\langle E \rangle = \frac{1}{2}k_BT$ per ... degree of freedom

E_T spectral emittance of blackbody: energy being emitted per volume btw f & $f+df$. Find out energy density. Measure spectral intensity ... how much energy ...

... can't distinguish being in motion at ... c is constant & equal isotropically for all observers; ... c moves isotropically ... observer sees simultaneity ... observer claims the rider would see front of train because rider is moving ... lightning first

The person on train observes front of train getting ...

$x' = \gamma(x - vt)$; $x = \gamma(x' + vt')$; $t' = \gamma\left(t - \frac{vx}{c^2}\right)$; $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$; $\gamma = \left(\sqrt{1-\beta^2}\right)^{-1}$, $\beta = \frac{v}{c}$

$\Delta x' = \gamma(\Delta x - v\Delta t) = 0$; $L_0 = L\gamma$ proper length: length in its rest frame; $\Delta t = \gamma \Delta t_0$ proper time: moving clock, observer sees rest... slower & is not contracted; he sees moving objects contracted

Post. ...: state of particle is completely represented by wavefunction; observable quantities are represented as operators

deBroglie Post.: particles can behave like waves but also like particles depending on the experiment you perform — wave-particle duality

Correspondence Principle: there is a limit for which classical mech & QM give the same answer; the quantum sol'n approaches its classical limit

Bragg's Law: when x-rays scattered from crystal lattice at conditions that $\theta_i = \theta_{scattered}$: $n\lambda = 2d\sin\theta$; allows to calc details about crystal structure, λ of x-rays incident upon crystal

... energy state $n = 1,2,3,\ldots$; l = angular momentum # → orbital's shape $= (s, p, d, f, \ldots)$; orbital is region where e^- is 90% likely to be found. $l = 0, 1, 2, \ldots n-1$; m_l = magnetic Q # relates to angular # by $L_z = m_l\hbar$ & describes magnetic prin. of atom $m_l = -l, -l+1, \ldots 0, \ldots l-1, l$; m_s spin of a particle $\pm\frac{1}{2}$

... theory predicts that for given value of l, only certain orientations of the angular momentum vector in space are allowed — space quantization ⇒ $\frac{m_l}{\sqrt{l(l+1)}} = \frac{L_z}{|\vec{L}|}$, $L^2 = l(l+1)\hbar^2$, $L_z = m_l\hbar$ possible values of angular momentum

Spin-Coupling: the e^- has own ref. view, feels like nucleus is circling it ∴ creating its own B field & creating Q # m_s

Scattered e^- produces longer λ than original x-ray; larger λ means smaller f, lower E — Compton Effect

Stern-Gerlach Exp.: showed angular momentum can be quantized ... through it would ...

Davisson-Germer Exp.: deBroglie Hypoth. ...

Michaelson-Morley Exp. tried to find aether: Earth moves quickly if aether exists. Earth moving through "wind" = same way you see wind outside moving car; aether seems like moving substance on Earth. They measured speed of light travelling in different directions. By looking @ interference patterns in recombined light, changes due to aether wind ...

Twin Paradox: I think you will be younger after your trip, but you think I'll be younger since in your perspective it was me on Earth who was moving.

Sol'n:

Since you are moving, what you think of as forward dir. of time is rotated relative to my perception of space time.

motion of how time passes changed & so you have gaps in it — what should happen: instantaneous change; from your perspective it does take 8 sec. Since you changed ... when you turned around to come back, your ...

me: 3.6 sec; you: 8 sec

eqs: $\Delta t' = \gamma(\Delta t)$; $\Delta x' = \gamma(\Delta x)$ twin paradox; should be: $\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)$; $\Delta x' = \gamma(\Delta x - v\Delta t)$

... should have been detected ... magnets ... B field ... mirror ... light source, semi-silvered mirror, detector

... shift observed in the freq. of an EM wave when source of wave is moving w/ respect to observer ⇒ Doppler Effect. Based on if source is moving towards or away. wave emitted $d = c\Delta t$, source $d = v\Delta t$ ⇒ $\lambda = \frac{c\Delta t - v\Delta t}{N}$ waves emitted ⇒ $f = \frac{c}{\lambda} = \frac{cN}{(c-v)\Delta t} = \frac{1}{1-\beta}\frac{N}{\Delta t}$; $\Delta t = \gamma\Delta t$; $f = \frac{f_0}{1-\beta}\frac{1}{\gamma}$

Derive $\vec{v}$ transform: $u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - \frac{v\,dx}{c^2}\right)}$; $u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v\,dx}{c^2}\right)}$ divide num & den by dt

$u'_x = \frac{dx - v\,dt}{dt - \frac{v\,dx}{c^2}} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$; $u'_y = \frac{\frac{dy}{dt}}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_y/\gamma}{1 - \frac{vu_x}{c^2}}$

sub in $u_x = \frac{dx}{dt}$; u'_z is same

$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$; inverse: $u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$; $u_y = \frac{u'_y/\gamma}{1 + \frac{vu'_x}{c^2}}$

$\vec{u} = (u_x, u_y, u_z)$; $\vec{u}' = (u'_x, u'_y, u'_z)$ of object in frame F; F' moves w/ $\vec{v} = (v, 0, 0)$ relative to F; the $\vec{u}'$ of object

Four Vector: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$x'^0 = \gamma(x^0 - \beta x^1) = \Lambda^0{}_0 x^0 + \Lambda^0{}_1 x^1$

$x'^1 = \gamma(x^1 - \beta x^0) = \Lambda^1{}_0 x^0 + \Lambda^1{}_1 x^1$

$x'^2 = x^2 = \Lambda^2{}_2 x^2$; $x'^3 = x^3 = \Lambda^3{}_3 x^3$

$x^\mu = (ct, x, y, z)$; $\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

length of vector: $|x^\mu|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$; $|x'^\mu|^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$

$|x^\mu|^2 = g_{00}x^0x^0 + g_{11}x^1x^1 + g_{22}x^2x^2 + g_{33}x^3x^3$

$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$; $|x|^2 = g_{\mu\nu}x^\mu x^\nu$; $|x'|^2 = g_{\mu\nu}x'^\mu x'^\nu$

To measure any value, apply operator; avg. momentum: $\hat{p}\psi = p\psi$ ⇒ $\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\,\hat{p}\,\psi(x)\,dx = -i\hbar\int \psi^* \frac{d\psi}{dx}dx$, operator to ψ then mult. by ψ^*, integrate

$\langle K \rangle = \frac{1}{2m}\langle p^2 \rangle = \frac{1}{2m}\int_{-\infty}^{\infty} \psi^*(x)\,\hat{p}^2\,\psi(x)\,dx = \frac{-\hbar^2}{2m}\int \psi^*(x)\frac{d^2}{dx^2}\psi(x)\,dx$

$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x)\,x\,\psi(x)\,dx$

Uncertainty Principle: $\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$; $(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$; $(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle = \langle p^2 \rangle$, momentum $= 0$

$n=0$ state $\langle x \rangle = 0$; $(\Delta x)^2 = \langle x^2 \rangle$ only then is ...; $(\Delta p)^2 = \langle p^2 \rangle = 2m\langle K \rangle = 2m\frac{\hbar^2}{4m\sigma^2} = \frac{\hbar^2}{2\sigma^2}$ ⇒ $\Delta p = \ldots$

$(\Delta x)^2 = \langle x^2 \rangle = \int A e^{-x^2/2\sigma^2} x^2 A e^{-x^2/2\sigma^2}dx = A^2\int x^2 e^{-x^2/\sigma^2}dx$

$= \frac{1}{2}A^2\sqrt{\pi}\,\sigma^3 = \frac{1}{2}\sigma^2$ ⇒ $\Delta x = \frac{\sigma}{\sqrt{2}}$

$\Delta t \cdot \Delta E \geq \frac{\hbar}{2}$; $\sigma_x \sigma_{p_x} \geq \frac{\hbar}{4}$

Photoelectric effect: light was shun onto a metal plate; it was expected to behave like a wave, but e^- emitted from the metallic surface ...

Simultaneity: 2 events that occur simultaneously to one observer may occur at different times relative to another observer

$E > V_0$ step: Reflection $= \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$; Trans $= \frac{4k_1k_2}{(k_1+k_2)^2}$; $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$; $k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$

Wein's Displacement Law: $\frac{hf}{k_BT} = 2.8214$

$\frac{1}{\lambda} = R_H\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$

orbital ang. momentum, magnetic Q #: $\mu_s = \frac{-e}{2m}g_s S$; $S^2 = s(s+1)\hbar^2$, e^- $s = \frac{1}{2}$; $S_z = m_s\hbar$, m_s ranges $s, s-1, \ldots, -s$; e^- $m_s = \pm\frac{1}{2}$

$L\psi_{nlm} = \sqrt{l(l+1)}\,\hbar\,\psi_{nlm}$; $L_z\psi_{nlm} = m_l\hbar\,\psi_{nlm}$; $|m_l| \leq l$

Splitting of energy levels = Zeeman effect; $j = |l+s| \ldots |l-s|$; $J_z = m_j\hbar$; $J = \sqrt{j(j+1)}\,\hbar$; $j = l + \frac{1}{2}$

$KE = hf - \phi$ work func (energy required to rip off e^-, binding energy); $E_2 - E_1 = hf$; max KE; eV_0 (photoelectric effect)

$T = \frac{4k_1k_2}{(k_1+k_2)^2}$; $T + R = 1$

Energy operator: $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V$

avg. value of an observable: $\langle Q \rangle = \int_a^b \psi^*\,\hat{Q}\,\psi(x)\,dx$

$E_2 - \frac{\hbar^2}{2m}$ ... Δm ... ± 1 ... Δl all ...

Derive commutator $[a,a^\dagger]$

$[a,a^\dagger] = \left[\sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right), \sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{ip}{m\omega}\right)\right]$

$= \frac{m\omega}{2\hbar}\left[\left(x+\frac{ip}{m\omega}\right),\left(x-\frac{ip}{m\omega}\right)\right]$; use $[x,p]=i\hbar$

$= \frac{m\omega}{2\hbar}\left\{[x,x] - \frac{i}{m\omega}[x,p] + \frac{i}{m\omega}[p,x] + \frac{1}{m^2\omega^2}[p,p]\right\}$; since $[x,x]=[p,p]=0$

$= \frac{m\omega}{2\hbar}\left\{-\frac{i}{m\omega}[x,p] + \frac{i}{m\omega}[p,x]\right\}$

$= \frac{i}{2\hbar}\left\{-[x,p]+[p,x]\right\} = \frac{i}{2\hbar}\left\{-[x,p]-[x,p]\right\}$

$= \frac{-i}{\hbar}[x,p] = \frac{-i}{\hbar}(i\hbar) = 1$ i.e. $[a,a^\dagger]=1$

Show $\langle n|x|n\rangle = \langle n|p|n\rangle = 0$

$x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)$

$p = -i\sqrt{\frac{m\hbar\omega}{2}}(a-a^\dagger)$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$= \frac{\hbar}{2m\omega}\left(n+\frac{1}{2}\right)$ $\langle n|x|n\rangle^2$

$\langle p^2\rangle = \hbar m\omega\left(n+\frac{1}{2}\right)$; $\Delta x\Delta p \geq \frac{\hbar}{2}$; $[p,x^n] = -i\hbar n x^{n-1}$

Osc. in momentum space: mom. space e.function obtained through coord. space: $x\to p$, $m\omega \to \frac{1}{m\omega}$; $\psi(p) = \frac{1}{(m\omega\pi\hbar)^{1/4}}e^{-\frac{p^2}{2m\omega\hbar}}$

$\langle n|X^k|n\rangle = \langle x^k\rangle$; use this & $a^\dagger$, a; $(a+a^\dagger)^k \to 2^k$ terms; count up & down; use $\delta_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$\frac{\hbar}{2m\omega}(n+\frac{1}{2})$; $\langle n|x^4|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^2\langle n|(a+a^\dagger)^4|n\rangle$

Aug. 2 down will get same n state to $\langle n|n\rangle = 1$ (i.e. terms that go to zero)

Changing basis: Harmonic osc.: $H = a^\dagger a + 1/2$; $\hat{p},\hat{x}$ in energy basis

$x = \sqrt{\hbar/2m\omega}\,(a+a^\dagger)$

$p = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)$

$\Delta x = \sqrt{\frac{\hbar}{m\omega}(n+\frac{1}{2})}$

$\Delta p = \sqrt{\hbar m\omega(n+\frac{1}{2})}$

$\langle 3|[x,p]|2\rangle = 0$; $\langle 3|\ |3\rangle = \frac{i\hbar}{2}\times 0$

$\langle[\rho,H]\rangle = \langle\psi|[\ ]|\psi\rangle = 0$

$\langle\hat{p}\rangle = 0$ by Ehrenfest's Thm.

$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$

$e^{iA} = \sum_{n=0}^{\infty}\frac{(iA)^n}{n!}$; Prove $U = e^{iH}$ is unitary: $U^\dagger = [e^{iH}]^\dagger = e^{-iH^\dagger}$; since H is Hermitian $\Rightarrow e^{-iH}$; $\therefore UU^\dagger = e^{+iH}e^{-iH} = 1$

$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$; normalized $\psi(p)$ in position rep.

Fourier Transform: $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p)e^{ipx/\hbar}dp$ $\Rightarrow$ position space

$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int\psi(x)e^{-ipx/\hbar}dx$ momentum space

$\psi_n(u) = H_n(u)e^{-u^2/2}$; $u = \sqrt{\frac{m\omega}{\hbar}}x$

$\psi_n(x) = A_n H_n(x)e^{-m\omega x^2/2\hbar}$ in position coord.

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

Prob. density: $|\psi|^2$

Hermitian: $H = H^\dagger$

$\psi(x) = \langle x|\psi\rangle$; $\langle x|\mathbb{I}|\psi\rangle = \int\langle x|p\rangle\langle p|\psi\rangle dp$

$|\psi\rangle = \int\psi(x)dx$; $\langle\psi|A|\psi\rangle = \int\psi^*\psi\,dx$; $\langle\psi|A|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x)A\psi(x)dx$

$\psi(x,0) = \frac{1}{\sqrt{3}}\psi_0(x) + \frac{1}{\sqrt{3}}\psi_1(x)$ ...; Particle in the box; $E = \hbar\omega$; $\psi(x,t) = \sum c_n\phi_n(x)e^{-iE_nt/\hbar}$; find $\langle x\rangle$

$|\psi(t)\rangle = e^{-iE_it/\hbar}|\psi(t_0)\rangle$; Spec. Decomp.: $A = \sum\lambda_i|a_i\rangle\langle a_i|$

$\psi_n(x) = \sqrt{2}\pi^{-1/4}\left(\frac{m\omega}{\hbar}\right)^{3/4}xe^{-\frac{m\omega x^2}{2\hbar}}$

$\int_{-\infty}^{\infty}z^{2n}e^{-z^2}dz$; $z^ne^{-z^2}dz = \frac{\sqrt{\pi}\,1\cdot3\cdot5\cdots(2n-1)}{2^n}$; $=0$ (n odd)

$UU^\dagger = U^\dagger U = \mathbb{I}$ (unitary); $\langle\omega|T|u\rangle = \langle u|T|\omega\rangle$; $T_{ij} = \langle i|T|j\rangle$ row column

Show $H|a\rangle = (E_n - \hbar\omega)a|n\rangle$ ... similar w/ $[H,a^\dagger]$

Show $a|n\rangle$ is e.vect of H w/ e.val $E_n - \hbar\omega$ & $a^\dagger|n\rangle$ " " " " " $E_n + \hbar\omega$

$[H,a] = Ha - aH \Rightarrow Ha = aH - \hbar\omega a$

$-\hbar\omega a$: $[H,a^\dagger] = \hbar\omega a^\dagger$

$\Rightarrow H(a|n\rangle) = (Ha)|n\rangle$

$= (aH - \hbar\omega a)|n\rangle$

$= aH|n\rangle - \hbar\omega a|n\rangle$

$= a[\hbar\omega(n+\frac{1}{2})|n\rangle] - \hbar\omega a|n\rangle$

$= (E_n - \hbar\omega)a|n\rangle$

at $t=0$: $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{6}}|2\rangle$

energy measured, what values & prob? ii) avg value of energy $\langle H\rangle$. iii) find forms of $\Phi_i(x)$, basis funct. & write form of wavefunction at t.

basis state energy prob: $\{|0\rangle, \frac{\hbar\omega}{2}, \frac{1}{3}\}, \{|1\rangle, \frac{3\hbar\omega}{2}, \frac{1}{3}\}, \{|2\rangle, \frac{5\hbar\omega}{2}, \frac{1}{6}\}$

$\langle H\rangle = \sum P_iE_i = \frac{7\hbar\omega}{6}$; nth state wavefunction of Harmonic Osc.

$\Phi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^nn!}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(\frac{-m\omega}{2\hbar}x^2\right)$

$\Phi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}(\ )$; $\Phi_1(x) = \langle x|1\rangle$; $\Phi_2(x) = \langle x|2\rangle$

at $t=0$: $\psi(x,0) = \frac{1}{\sqrt{3}}\Phi_0(x) + \frac{1}{\sqrt{3}}\Phi_1(x) + \frac{1}{\sqrt{6}}\Phi_2(x)$

at t: $e^{-iE_nt/\hbar}$

Show $\langle a|A|a\rangle = \langle a|(A|a\rangle) = \lambda\langle a|a\rangle$; $\langle a|A|a\rangle = \lambda^*\langle a|a\rangle \Rightarrow 0 = (\lambda - \lambda^*)\langle a|a\rangle \Rightarrow \lambda = \lambda^*$ (Hermitian)

Show Hermitian operators have e.values real

$X|x\rangle = x_i|x\rangle$

Show e.vects of diff. e.vals are orthonormal: let $A|\lambda_1\rangle = a_1|\lambda_1\rangle$ & $A|\lambda_2\rangle = a_2|\lambda_2\rangle$ ②; $\langle\lambda_1|A = a_1^*\langle\lambda_1|$ ① mult. by ① by $|\lambda_2\rangle$; & subtract: mult ② by $\langle\lambda_1|$ & s.: $\langle\lambda_1|A|\lambda_2\rangle - \langle\lambda_1|A|\lambda_2\rangle = 0 = (a_1 - a_2)\langle\lambda_1|\lambda_2\rangle$; since $a_1 \neq a_2 \Rightarrow \langle\lambda_1|\lambda_2\rangle = 0$

$\langle U_1|\psi\rangle$, $\langle U_2|\psi\rangle$

If $[A,B] = 0 \Rightarrow$ they share common eigenvectors: let $A|a\rangle = a|a\rangle \Rightarrow BA|a\rangle = B(a|a\rangle)$; since $[A,B]=0 \Rightarrow A(B|a\rangle) = a(B|a\rangle)$; So $B|a\rangle$ is also an e.state of $\hat{A}$ w/ e.value a. Since $\nexists$ no degeneracy, each e.value is unique eig.state so $B|a\rangle$ must be same scalar multiple of $|a\rangle$, i.e. $A(B|a\rangle) = \lambda|a\rangle = a(b|a\rangle)$; So $|a\rangle$ is also an e.state B i.e. $B|a\rangle = b|a\rangle$

Prove $[X,P] = i\hbar \to$ apply $[X,P]\psi(x) = [XP - PX]\psi(x) = XP\psi(x) - PX\psi(x)$

Show that $[N,a] = -a$, $[N,a^\dagger] = a^\dagger$

use: $[A,B] = -[B,A]$, $[A,BC] = [A,B]C + B[A,C]$

$[N,a] = [a^\dagger a, a] = -[a, a^\dagger a] = -[a,a^\dagger]a - a^\dagger[a,a]$

$= -[a,a^\dagger]a = -a$; repeat for $[N,a^\dagger]$

Use recursion relation for Hermite polynomials to find $\langle x\rangle$ & $\langle p\rangle$ for ground state H. osc. in coordinate representation:

$\psi_n(x) = \frac{1}{\sqrt{2^nn!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega x^2}{2\hbar}}$

$\langle x\rangle = \int_{-\infty}^{\infty}\psi_n^*(x)x\psi(x)dx = \frac{1}{2^nn!}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\infty}^{\infty}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)(x)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega x^2}{\hbar}}dx$

use: (recursion formula): $H_{n+1}(u) = 2uH_n(u) - 2nH_{n-1}(u)$; rewrite integral $\to uH_n(u) = \frac{1}{2}H_{n+1}(u) + nH_{n-1}(u)$

Hermite polynomial satisfies: $\int_{-\infty}^{\infty}H_m(u)H_n(u)e^{-u^2}du = \sqrt{\pi}\,2^nn!\,\delta_{mn}$; $m\neq n \Rightarrow 0$

$p = i\hbar\frac{\partial}{\partial x}$ (in coord. rep.)

do Hermite integral using $\frac{dH_n}{du} = 2nH_{n-1}(u)$; $\langle x\rangle = 0$ in grnd state (or any sol. $m\neq n$)

$\langle\psi|P|\psi\rangle = \langle p\rangle = \iint\langle\psi|x'\rangle\langle x'|P|x''\rangle\langle x''|\psi\rangle dx'dx'' = \iint\psi^*(x')\left[\frac{-i\partial}{\hbar\partial x''}\psi(x'')\delta(x'-x'')\right]dx'dx'' = \frac{-i}{\hbar}\int\psi^*(x')\frac{\partial}{\partial x'}\psi(x')dx'$

translation operator: $T(\epsilon)|x\rangle = e^{i\epsilon g(x)}|x+\epsilon\rangle$

$e^{i\theta} + e^{-i\theta}$ over 2 $= \cos\theta$; $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

$a|n\rangle$ into p-basis to get $\psi_0(p)$; $\langle p|a|n\rangle = 0$ plug in; $\hat{X} = i\hbar\frac{\partial}{\partial p}$

$\begin{pmatrix}1/2 & 0 & 0\\ 0 & 3/2 & 0\\ 0 & 0 & 5/2\end{pmatrix}$; $|\psi\rangle = \sum|a_i\rangle\langle a_i|\psi\rangle$; $= \sum\langle a_i|\psi\rangle|a_i\rangle = \sum|r|\alpha\rangle$

Rows & columns of unitary matrices form an orthonormal set $(\text{row})_m^* \cdot (\text{row})_n = 0$

Initial state $|\psi\rangle_{old}$ can be represented in energy eigenbasis by using projection operators: $|\psi\rangle_{new} = (\sum|E_i\rangle\langle E_i|)|\psi\rangle_{old}$. So, given H: $|\psi\rangle = c_1|a_1\rangle + c_2|a_2\rangle \Rightarrow \det|H - \lambda I| = 0 \to |E_1\rangle = \frac{1}{\sqrt{2}}\binom{1}{1}$, $|E_2\rangle = \frac{1}{\sqrt{2}}\binom{1}{-1}$

$\Rightarrow |E_1\rangle = \frac{1}{\sqrt{2}}(a|a_1\rangle + b|a_2\rangle)$ & $|E_2\rangle = \frac{1}{\sqrt{2}}(c|a_1\rangle + d|a_2\rangle)$

$|\psi\rangle = a|+\rangle + b|-\rangle \Rightarrow |\langle\pm|\psi\rangle|^2$; $\left|\frac{1}{\sqrt{2}}\langle +| \pm \langle -|\,|\psi\rangle\right|^2$

Spin prob. $|\pm x\rangle$ ups^2

General: 2×2 spin $\frac{1}{2}$: $|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\rangle$; $|\pm\hat{n}\rangle$-basis

$|+n\rangle = |1\rangle = \mathbb{I}|1\rangle = |1\rangle_x\langle 1|1\rangle + |-1\rangle_x\langle -1|1\rangle$

$\langle x|T(a)|\psi\rangle = \psi(x) - \frac{d\psi}{dx}a + \frac{d^2\psi}{dx^2}\frac{a^2}{2!} + \ldots$

$e^{-ax} = \lim_{N\to\infty}\left(1 - \frac{ax}{N}\right)^N$

$T(a) = e^{-iaP/\hbar}$

$P_{+\hbar} = |\langle 1|\psi_+\rangle|^2 = |c_1|^2$

$S = \begin{pmatrix}\langle +z|+n\rangle & \langle +z|-n\rangle\\ \langle -z|+n\rangle & \langle -z|-n\rangle\end{pmatrix}$

Matrix representation of A in S basis: $\hat{A}_n = \begin{pmatrix}\langle +n|A|+n\rangle & \langle +n|A|-n\rangle\\ \langle -n|A|+n\rangle & \langle -n|A|-n\rangle\end{pmatrix}$; $A_n = S^\dagger A S$; $S^{\dagger}[S]^*$

Change of basis ($z\to n$) operator: $|\psi\rangle_n = S^\dagger|\psi\rangle_z$; $S_n\langle\psi| = {}_z\langle\psi|S$

$\langle\psi|A|\psi\rangle$ n-basis to n-basis: ${}_n\langle\psi|S^\dagger A_z S|\psi\rangle_n$

Change of basis ($z\to n$): $|\psi\rangle_z = (1_z)$; $[|+n\rangle\langle +n| + |-n\rangle\langle -n|](|+z\rangle\langle +z| + |-z\rangle\langle -z|)$

$= (|+n\rangle\langle +n|+z\rangle\langle +z|\psi\rangle + \ldots)$

Matrix representation: $\begin{pmatrix}\langle +n|\psi\rangle\\ \langle -n|\psi\rangle\end{pmatrix} = \begin{pmatrix}\langle +n|+z\rangle & \langle +n|-z\rangle\\ \langle -n|+z\rangle & \langle -n|-z\rangle\end{pmatrix}\begin{pmatrix}\langle +z|\psi\rangle\\ \langle -z|\psi\rangle\end{pmatrix}$; $|\psi\rangle_n = S^\dagger|\psi\rangle_z$

So for states: $|b'\rangle = \sum|a''\rangle\langle a''|b'\rangle$; w/ $U_b = \sum|b_m\rangle\langle a_m|$ ... i.e. $|b_i\rangle = U_{ij}|a_i\rangle$

this can be viewed as matrix U

$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$; $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$; $S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

$|\pm\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$; $|\pm\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle)$

$|1\rangle_x = \frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}$; $|1\rangle_y = \frac{1}{2}(|1\rangle + \sqrt{2}i|0\rangle - |-1\rangle)$; $|-1\rangle_y = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}i\\ -1\end{pmatrix}$ in z basis

Finding each projection of $|1\rangle_y$ prob of measurement along z-axis?

$P_x = |\langle 1|\ 1_y\rangle|^2$; $P_0 = |\langle\psi|\ \rangle|^2$

Wigner Quest. on E1: two state $|1\rangle$ & $|2\rangle$; $P_1 = \langle 1|A_1\rangle$; $P_2 = \langle 2|A_2\rangle$; $\langle a|\lambda_a\rangle$

$|R\rangle$: $|H_2\rangle = a|+\rangle + b|-\rangle$; $\Rightarrow a = \frac{1}{\sqrt{3}}$, $b = \frac{\sqrt{2}}{\sqrt{3}}e^{i\phi}$

$= \left[\frac{1}{\sqrt{3}}(\langle +| + \langle -|)\right]$; $= |\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}e^{i\phi}|^2$

Given $P_0 = \frac{1}{2}$, $P_{a_1} = \frac{3}{4}$; $P_{a_2} = \frac{1}{4}$; Find $|\psi_0\rangle$

$\frac{1}{2}|1 + \cos\phi|$; $\phi = \frac{\pi}{3}$

Given $H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|)$; $|\alpha\rangle = \begin{pmatrix}0 & \Delta\\ \Delta & 0\end{pmatrix}$

Apply time evolution: $|\alpha_t\rangle = U|\alpha\rangle = e^{-iHt/\hbar}|\alpha\rangle$

Probability of finding in left: $P = |\langle L|\alpha_t\rangle|^2$; Rewrite in terms of energy basis:

e-vals: $\lambda = \pm\Delta$

$\Rightarrow |E_1\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$; $|E_2\rangle = \frac{1}{\sqrt{2}}(|R\rangle - |L\rangle)$

$\Rightarrow |L\rangle = \frac{\sqrt{2}}{2}|E_1\rangle - |R\rangle$; & $|R\rangle = \sqrt{2}|E_2\rangle + |L\rangle$ plug in

$\Rightarrow |R\rangle = \frac{\sqrt{2}}{2}(|E_1\rangle + |E_2\rangle)$; $|L\rangle = \frac{\sqrt{2}}{2}(|E_1\rangle - |E_2\rangle)$

$\hat{A}$ has 2 normalized eigenstates $|\psi_1\rangle$, $|\psi_2\rangle$ & e.values a_1, a_2

$\hat{B}$ has " " " $|\Phi_1\rangle$, $|\Phi_2\rangle$ & e.values b_1, b_2

$|\psi_1\rangle = \frac{1}{\sqrt{13}}(2|\Phi_1\rangle + 3|\Phi_2\rangle)$; $|\psi_2\rangle = \frac{1}{\sqrt{13}}(3|\Phi_1\rangle - 2|\Phi_2\rangle)$

$\hat{A}$ measured $\to a_1$. If B is measured & then A again, prob. of getting a_1? Probability of getting a_1 as well

A measured $\to |\langle\Phi_1|\psi_1\rangle|^2 = P_{b_1}$; $|\psi\rangle$ state collapsed to one of these (i.e. b_1)

B measurement ... a_1 recurs collapse to a_2 instead

A measured again $\to$ if B measured gave $|\Phi_1\rangle$ (i.e. b_1) $\Rightarrow |\langle\psi_1|\Phi_1\rangle|^2$

so total prob of getting a_1: $P_{b_1} + P_{b_2}$; if " " " $|\Phi_2\rangle$ (i.e. b_2) $\to |\langle\psi_1|\Phi_2\rangle|^2 = P_{a_2}$; $= \frac{82}{169}$

$\epsilon_{132} = -1$; $\epsilon_{123} = 1$

$\vec{a}\times(\vec{b}\times\vec{c}) = \epsilon_{ijk}a_j\epsilon_{klm}b_lc_m = \epsilon_{ijk}\epsilon_{klm}a_jb_lc_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_jb_lc_m = (\sum a_jc_j)b_i - (\sum a_jb_j)c_i$

In the ... place ...

$[L_i, L_j] = i\hbar L_k$; $[A, B+C] = [A,B] + [A,C]$; $[A, BC] = [A,B]C + B[A,C]$

$L_x = yp_z - zp_y$; $L_y = zp_x - xp_z$; $L_z = xp_y - yp_x$ — use first, use 2nd to show

$L^2 = l_x^2 + l_y^2 + l_z^2$; $[L_z, L^2] = [L_z, L_x^2] + [L_z, L_y^2] + [L_z, L_z^2]$

$(\Delta A)^2(\Delta B)^2 \geq \left|\frac{\langle [A,B] \rangle}{2i}\right|^2$

System of $l=1$ is in state: $|\psi\rangle = \frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{2}}|-1\rangle$ Find $\langle L_y \rangle$

eval: $\{-\hbar, 0, \hbar\}$

$J_+ = \sqrt{2}\hbar \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix}$; $J_- = \sqrt{2}\hbar \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix}$

$J_z = \hbar \begin{pmatrix} 1&0&0\\0&0&0\\0&0&-1 \end{pmatrix}$

$J_x = \frac{J_+ + J_-}{2} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0&1&0\\1&0&1\\0&1&0 \end{pmatrix}$ J_x in J_z basis

Wavefunction for $0 \leq x \leq a$ in ground state is $\psi(x) = A\sin\left(\frac{\pi x}{a}\right)$. Find A & prob particle is between $\frac{a}{2} \leq x \leq \frac{3a}{4}$

$\int_{-\infty}^{\infty} |\psi|^2 dx = 1 = \int_0^a A^2 \sin^2\left(\frac{\pi x}{a}\right) dx = A^2 \int_0^a \frac{1 - \cos\left(2\left(\frac{\pi x}{a}\right)\right)}{2} dx$

$= \frac{A^2}{2}\int_0^a dx - \frac{A^2}{2}\int_0^a \cos\left(2\left(\frac{\pi x}{a}\right)\right) dx \Rightarrow \frac{A^2}{2} a = 1$

Prob: $P\left(\frac{a}{2} \leq x \leq \frac{3a}{4}\right) = \int_{a/2}^{3a/4} |\psi(x)|^2 dx$ (same integral)

Similarity transformation allows to put matrix into diagonal form → elements along diag are eigenvalues. D is diagonal & C & D represent same operator w.r.t two different bases. $D = S^{-1}CS$

Process: 1) Find e.vals & vectors of C. 2) Normalize e.vectors. 3) Form S^{-1} by using eigenvectors of C as columns of C

In QM: $U^\dagger = U^{-1} \Rightarrow \hat{D} = U^\dagger \hat{H} U$ Unitary Transform

Diagonalize: $\det|A - \lambda I| = 0$

Projection operator: $P = P^\dagger$, $P = P^2$

Taylor series for a func. of an operator; $[A, B^n] = n[A,B]B^{n-1}$

$F(A) = \sum_{n=0}^{\infty} a_n A^n$

$P_m = \sum_{i=1}^{n} |u_i\rangle\langle u_i|$; $I = \sum |\ldots$

$\langle x|p\rangle = \langle p|x\rangle^* = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

$\psi(x) = \langle x|\psi\rangle = \langle x|I|\psi\rangle = \int \langle x|p\rangle\langle p|\psi\rangle dp = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar}\bar{\psi}(p) dp$

Dirac Delta: $\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$

F.T of Dirac Delta: $\frac{1}{\sqrt{2\pi}} \int \delta(x) e^{-ikx} dx = \ldots$

$\int_{-\infty}^{\infty} x e^{-ax^2} dx = 0$; $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = 2\int_0^\infty x^2 e^{-ax^2} dx$

Particle in 1D box $0 \leq x \leq a$ in state $\psi(x) = \frac{1}{\sqrt{10a}}\sin\left(\frac{\pi x}{a}\right) + A\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right) + \frac{3}{\sqrt{5a}}\sin\left(\frac{3\pi x}{a}\right)$. Find A & possible energies & prob.

Energy measured to be $\frac{2\pi^2\hbar^2}{ma^2}$, what is state of system immediately post-measure?

Basis functions for 1D box: $\Phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

$\psi(x) = \frac{1}{\sqrt{20}}\Phi_1(x) + A\Phi_2(x) + \frac{3}{\sqrt{10}}\Phi_3(x)$

Compute inner product: $\frac{1}{20} + A^2 + \frac{9}{10} = \frac{19}{20} + A^2 = 1 \Rightarrow A = \frac{1}{\sqrt{20}}$

For box of length $2a \to E_n = \frac{n^2\pi^2\hbar^2}{8ma^2}$

$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

$\int_{-\infty}^{\infty} z^{2m} e^{-z^2} dz = \sqrt{\pi}\frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m}$

$\langle \psi(x), \psi(x) \rangle = 1$

If $\psi(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{3\pi x}{a}\right)$ what's prob. particle is found in $\Phi_n(x)$: $\langle \Phi_n(x), \psi(x) \rangle = \int \Phi_n^*(x)\psi(x) dx = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right) dx$

$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\int \cos\left(\frac{\pi}{a}\left(\frac{n}{2} - 3\right)x\right) dx = \frac{a}{\pi\left(\frac{n}{2}-3\right)}\sin\left(\pi\left(\frac{n}{2}-3\right)\right)$

Find ground state is $n = 1$; $\langle x \rangle$, $\langle p \rangle$ if particle in 1D box $0 \leq x \leq a$ in ground state

$\psi_1(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) \Rightarrow \langle x \rangle = \int \psi_1^*(x)[x]\psi_1(x) dx = \frac{2}{a}\int_0^a x\sin^2\left(\frac{\pi x}{a}\right) dx$

$= \frac{2}{a}\int_0^a x\left(\frac{1 - \cos\left(\frac{2\pi x}{a}\right)}{2}\right) dx = \frac{2}{a}\int_0^a x\, dx - \frac{1}{a}\int_0^a x\cos\left(\frac{2\pi x}{a}\right) dx$

$\langle p \rangle = \int \psi_1^*(x)[\hat{p}]\psi_1(x) dx = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\left[-i\hbar\frac{d}{dx}\right]\left(\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\right) dx \Rightarrow \langle p \rangle = 0$ in ground state

Dispersion $\omega = \omega(k)$ relation: $\omega = \frac{\hbar k^2}{2m}$ free particle ; $v_p = \frac{\omega}{k}$; $v_g = \frac{d\omega}{dk}$

Given $\psi(x)$ find $\langle x \rangle$, $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

Show that $2\mathrm{Re}\langle\psi|\phi\rangle \leq \langle\psi|\psi\rangle + \langle\phi|\phi\rangle$

$\langle\psi-\phi|\psi-\phi\rangle \geq 0$; $|\psi\rangle - |\phi\rangle = |\psi-\phi\rangle$

$\langle\psi|\psi\rangle - \langle\psi|\phi\rangle - \langle\phi|\psi\rangle + \langle\phi|\phi\rangle \Rightarrow 0 \leq \ldots$

Cauchy-Schwartz: $|\langle\phi|\psi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle$

Triangle Inequality: use Cauchy-Schwartz

Are these linearly independent? $|a\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|b\rangle$, $|c\rangle$. To check: $c_1|a\rangle + c_2|b\rangle + c_3|c\rangle = 0$ — yes, linearly indep.

Gram-Schmidt Procedure: $\{|u_i\rangle\}$ is not orthonormal. To build orthonormal set $\{|v_i\rangle\}$: let $|w_1\rangle = |u_1\rangle$

$|w_2\rangle = |u_2\rangle - \frac{\langle w_1|u_2\rangle}{\langle w_1|w_1\rangle}|w_1\rangle$

$|w_3\rangle = |u_3\rangle - \frac{\langle w_1|u_3\rangle}{\langle w_1|w_1\rangle}|w_1\rangle - \frac{\langle w_2|u_3\rangle}{\langle w_2|w_2\rangle}|w_2\rangle$

$|v_1\rangle = \frac{|w_1\rangle}{\sqrt{\langle w_1|w_1\rangle}}$, $|v_2\rangle = \frac{|w_2\rangle}{\sqrt{\langle w_2|w_2\rangle}}$, $|v_3\rangle = \frac{|w_3\rangle}{\sqrt{\langle w_3|w_3\rangle}}$

$\mathrm{Tr}(AB) = \sum\langle u_i|AB|u_i\rangle$

Suppose Hamiltonian operator basis $|u_i\rangle$ i.e. $H = E|u_1\rangle\langle u_1| + E|u_2\rangle\langle u_2| + 2E|u_3\rangle\langle u_3| + 4E|u_4\rangle\langle u_4|$. Write projection operator P that projects onto subspace spanned by $\{|u_1\rangle, |u_2\rangle\}$? Energy measured: what results & probability? Energy measured found to be E, state of system after measurement?

Systems in state $|\psi\rangle = \ldots$

Possible results: $E, 2E, 4E$. If energy measured $= E$, state after measurement: $P_E|\psi\rangle = (|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|)|\psi\rangle$

$\hat{A} = \sum_i A_i |u_i\rangle\langle u_i|$

$\langle u|AB|v\rangle \to \lambda^*\langle u|B^\dagger A^\dagger|u\rangle$; $(AB)^\dagger = B^\dagger A^\dagger$

Hermitian operator: $\hat{F} = \hat{F}^\dagger$; anti-Hermitian: $A^\dagger = -A$

Check Hermiticity: $\langle g|Af\rangle = \langle Ag|f\rangle$

Unitary operator: $UU^\dagger = U^\dagger U = I$

Spectral Decomposition: representation of operator in terms of its e.values & projector

$A|a_i\rangle = \lambda|a_i\rangle \Rightarrow \hat{A} = \sum_i \lambda_i |a_i\rangle\langle a_i|$

$\int \psi^*(x)[\hat{p}\psi(x)] dx = \int (\psi(x)\hat{p})^* \psi(x) dx$

$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$; $\psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

Bound & Scattering: If particle in unbound region ($E > V$) $\Rightarrow \psi = Ae^{ikx} + Be^{-ikx}$

If particle's energy is less than barrier ($E < V$) $\Rightarrow \psi = Ae^{\beta x} + Be^{-\beta x}$

Find T: $k^2 = \frac{2mE}{\hbar^2}$, $\beta^2 = \frac{2m(V-E)}{\hbar^2}$

$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$; $\psi_{II} = Ce^{\beta x} + De^{-\beta x}$; $\psi_{III} = Ee^{ikx} + Fe^{-ikx}$

1) continuous at boundary; 2) 1st deriv. of ψ is continuous

Transmission: $T = \left|\frac{J_{transmitted}}{J_{incident}}\right|$; Reflection: $R = \left|\frac{J_{reflected}}{J_{incident}}\right|$

J = current density $= \frac{\hbar}{2mi}\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right)$

$\psi_I(x) = Ae^{\beta x} + Be^{-\beta x}$; $\psi_{II}(x) = Ce^{ikx} + De^{-ikx}$; $\psi_{III}(x) = Ee^{\beta x} + Fe^{-\beta x}$

$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$, $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi\rangle$

$\langle\psi(t)|A|\psi(t)\rangle = \langle\psi(0)|U^\dagger AU|\psi(0)\rangle$; $A(t) = U^\dagger A U$

$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$

$\frac{d}{dt}\langle A \rangle = \frac{1}{i\hbar}\langle[A,H]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$

$|\psi_s(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$; $e^{-iHt/\hbar} = \sum_{n=0}^{\infty}\left(\frac{-iHt}{\hbar}\right)^n\frac{1}{n!}$

$\langle A_s \rangle = \langle\psi_s(t)|A_s|\psi_s(t)\rangle$

Find $\Phi(p)$: $\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int\psi(x)e^{-ipx/\hbar} dx$

$[A, B] = AB - BA$; 2 operators commute iff they share a basis of eigenvectors. $[A,B] = 0$

$[A,B] = -[B,A]$; $\{A,B\} = AB + BA$; $[x,p] = i\hbar$; $[A, B+C] = [A,C] + [B,C]$

$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]$

$= yp_z zp_x - yp_z xp_z - zp_y zp_x + zp_y xp_z - zp_x yp_z + \ldots$

$= i\hbar(xp_y - yp_x) = i\hbar L_z$

$[p_x^2, x] = -2i\hbar p$; $[V(x), x] = 0$

Ehrenfest's Theorem: $H = \frac{p^2}{2m} + V(x)$; $\frac{d}{dt}\langle x \rangle = \ldots$

$\frac{d}{dt}\langle\psi|x|\psi\rangle = \frac{d}{dt}(\langle\psi|)x|\psi\rangle + \langle\psi|x\frac{d}{dt}(|\psi\rangle)$

$A|a_m^1\rangle = \lambda_m|a_m^1\rangle$; $A|a_m^2\rangle = \lambda_m|a_m^2\rangle$; $\Rightarrow \sum_{i=1}|\langle a_m^i|\psi\rangle|^2$

Prob of finding λ_m if degenerate

$H = \begin{pmatrix}\omega_1 & \omega_2\\ \omega_2 & \omega_1\end{pmatrix}$ basis: $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. Find eigvals & evecs, & time evolution of system by Schrodinger eq.

$H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$, $|\psi(0)\rangle = |0\rangle$

$\det(H - \lambda I) = 0 \Rightarrow \lambda^2 - 2\omega_1\lambda + (\omega_1^2 - \omega_2^2) = 0$

$\lambda_1 = \omega_1 + \omega_2$, $\lambda_2 = \omega_1 - \omega_2$

$\begin{pmatrix}\omega_1 & \omega_2\\ \omega_2 & \omega_1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = (\omega_1 + \omega_2)\begin{pmatrix}a\\b\end{pmatrix}$

$|A\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$; $|B\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\omega_1 & \omega_2\\ \omega_2 & \omega_1\end{pmatrix}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = \begin{pmatrix}\omega_1\alpha(t) + \omega_2\beta(t)\\ \omega_2\alpha(t) + \omega_1\beta(t)\end{pmatrix}$

$\omega_1\alpha + \omega_2\beta = i\hbar\dot{\alpha}$; $\omega_2\alpha + \omega_1\beta = i\hbar\dot{\beta}$

$(\omega_1 + \omega_2)(\alpha + \beta) = i\hbar(\dot{\alpha} + \dot{\beta})$; $\gamma = \alpha + \beta$; $\gamma = Ce^{\left(\frac{\omega_1 + \omega_2}{i\hbar}\right)t}$

$(\omega_1 - \omega_2)(\alpha - \beta) = i\hbar(\dot{\alpha} - \dot{\beta})$; $\delta = \alpha - \beta \Rightarrow \delta = De^{\left(\frac{\omega_1 - \omega_2}{i\hbar}\right)t}$

$\alpha = \frac{\gamma + \delta}{2} \Rightarrow \alpha = \frac{1}{2}\left[C\exp\left(\frac{\omega_1 + \omega_2}{i\hbar}t\right) + D\exp\left(\frac{\omega_1 - \omega_2}{i\hbar}t\right)\right]$

$|\psi(0)\rangle = \begin{pmatrix}\alpha(0)\\ \beta(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \alpha(0) = 1$; $C = D$; $\frac{C+D}{2} = 1$

$|\psi(t)\rangle = \begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$; $\alpha(t) = e^{-i\omega_1 t/\hbar}\cos\left(\frac{\omega_2 t}{\hbar}\right)$, $\beta(t) = ie^{-i\omega_1 t/\hbar}\sin\left(\frac{\omega_2 t}{\hbar}\right)$

$|\psi\rangle = \sum a_i|a_i\rangle$; $A|a_i\rangle = \lambda_i|a_i\rangle$; prob of getting λ_i: $|\langle a_i|\psi\rangle|^2 = |a_i|^2$

[illegible]

Thrm: If $\hat{A}$ is Hermitian
$\Rightarrow$ ① Eigenvalues are $\mathbb{R}$
$\Rightarrow$ ② e.vect. corresp to diff e.values are orthogonal

Pf) $\hat{A}|a\rangle = a|a\rangle$,
now $\langle a|\hat{A}^\dagger = a^*\langle a|$ } dual correspondence.
B/c $A^\dagger = \hat{A}$ so,
$A|a'\rangle = a'|a'\rangle$ & $\langle a''|A = a''^*\langle a''|$
$\Rightarrow \langle a''|$ B.s & $|a'\rangle$ on B.s
$\Rightarrow \langle a''|A|a'\rangle = a'\langle a''|a'\rangle$ & $\langle a''|A|a'\rangle = a''^*\langle a''|a'\rangle$
$\Rightarrow$ subtract: $(a' - a''^*)\langle a''|a'\rangle = 0$
$(a' - a''^*) = 0$ or $\langle a''|a'\rangle = 0$
$\Rightarrow a' = a''^*$ $\Rightarrow a' \neq a''$ by orthogonality
$\Rightarrow a' \in \mathbb{R}$

$\langle a''|a'\rangle = \delta_{a'',a'} = \begin{cases} 1 & \text{if } a''=a' \\ 0 & \text{if } a''\neq a' \end{cases}$

2 kets orthogonal if $\langle a''|a'\rangle = 0$

Thrm: If $\hat{A}, \hat{B}$ are compatible & e.vals of $\hat{A}$ are non-degenerate then matrix elements $\langle a''|B|a'\rangle$ are all diagonal (i.e. 2 operators that commute have same set of eigenkets!)

Pf) $\langle a''|(AB - BA)|a'\rangle = \langle a''|AB|a'\rangle - \langle a''|BA|a'\rangle$
$= a''\langle a''|B|a'\rangle - a'\langle a''|B|a'\rangle$
$= (a'' - a')\langle a''|B|a'\rangle = 0$, $\langle a''|B|a'\rangle = \delta_{a'a''}\langle a'|B|a'\rangle$

a' is a degenerate eigenvalue if $\exists$ more than one $|\psi_1\rangle, |\psi_2\rangle \dots |\psi_N\rangle$ s.t. $A|\psi_i\rangle = a|\psi_i\rangle$ & $|\psi_i\rangle$ being linearly independent

Thrm: $\hat{U}$ is unitary iff $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{I}$
Pf) let $\hat{U} = \sum_k |b^k\rangle\langle a^k| \Rightarrow \hat{U}^\dagger = \sum_k |a^k\rangle\langle b^k|$
$\Rightarrow UU^\dagger = (\ )(\ ) = \sum_{k,k'} |b^k\rangle\langle a^k|a^{k'}\rangle\langle b^{k'}|$
$= \sum_{k,k'} \delta_{k,k'}|b^k\rangle\langle b^{k'}|$, $\langle a^k|a^{k'}\rangle = \delta_{kk'}$
$= \sum_k |b^k\rangle\langle b^k| = \mathbb{I}$

$\vec{S}^2 \equiv S_x^2 + S_y^2 + S_z^2 = \left(\frac{3}{4}\right)\hbar^2$ • $[\vec{S}^2, S_i] = 0$
$= \vec{S}\cdot\vec{S}$

Diagonalization

$B|x\rangle = \lambda|x\rangle \rightarrow (B - \lambda I)|x\rangle = 0$
soln: $\det(B - \lambda I) = |B - \lambda I| = 0$
soln's will be $\lambda_1, \lambda_2 \dots \lambda_N$ (e.vals of B)
Ex: base kets $|1\rangle, |2\rangle$; $\hat{B} = |1\rangle\langle 2| + |2\rangle\langle 1|$
E.vals of B: $= \begin{pmatrix} \langle 1|1\rangle\langle 2|1\rangle & \dots \\ \dots & \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$|B - \lambda I| = 0$
$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$
E.vectors:
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} -\lambda x_1 + x_2 = 0 \\ x_1 - \lambda x_2 = 0 \end{cases}$
if $\lambda = 1 \Rightarrow x_1 = x_2$
e.vec is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, norm. const. $\frac{1}{\sqrt{a^2+a^2}}$
if $\lambda = -1$: $x_2 = -x_1$, e.vec: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Hermitian: $\hat{A} = \hat{A}^\dagger$; $\hat{A}|a\rangle \leftrightarrow \langle a|\hat{A}^\dagger$ "Hermitian adjoint of A"
$\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $\hat{A}^\dagger = (\hat{A}^*)^T = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$
"symmetric along diagonal"

$\hat{A}$: operator w/ eigenvalues $\{a', a'', a''', \dots\}$ ← possible results when you measure the observable $\hat{A}$. Eigenstate is the state corresponding to the eigenvalues $\{|a'\rangle, |a''\rangle, \dots\}$. E.value Eqn: $\hat{A}|a'\rangle = a'|a'\rangle$

Matrix Repr: $\hat{X} = \mathbb{I}\hat{X}\mathbb{I}$
$|\alpha\rangle = a|x\rangle + b|y\rangle$ in $\{|x\rangle, |y\rangle\}$ basis
$|\beta\rangle = c|x\rangle + d|y\rangle$
$= \sum_{a'}\sum_{a''} |a''\rangle\langle a''|\hat{X}|a'\rangle\langle a'|$ = $\mathbb{C}^\#$ operator — associative axiom
$= \sum_{a'}\sum_{a''} \underbrace{\langle a''|\hat{X}|a'\rangle}_{\text{row, col.}}(|a''\rangle\langle a'|)$ w/ $\langle a_1| = (1\,0\,0\dots 0)$, $\langle a_2| = (0\,1\,0\dots 0)$

$= \begin{bmatrix} \langle a^1|\hat{X}|a^1\rangle & \langle a^1|\hat{X}|a^2\rangle & \dots & \langle a^1|\hat{X}|a^N\rangle \\ \langle a^2|\hat{X}|a^1\rangle & \langle a^2|\hat{X}|a^2\rangle & \dots & \vdots \\ \vdots & & & \\ \langle a^N|\hat{X}|a^1\rangle & \dots & & \langle a^N|\hat{X}|a^N\rangle \end{bmatrix}$ & $|a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $|a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

arbitrary $\hat{X}$: $\langle a|\hat{X}|b\rangle = (\langle b|\hat{X}^\dagger|a\rangle)^*$
Hermitian $\hat{X}$: $\langle a|\hat{X}|b\rangle = (\ ") = \langle b|\hat{X}|a\rangle^*$ i.e. diagonal elements are $\mathbb{R}$

The Sqr Matrix is diagonal:
so $\hat{A} = \sum_{a''}\sum_{a'} |a''\rangle\langle a''|A'|a'\rangle\langle a'|$ $\langle a'|a''\rangle$ $\begin{pmatrix} \# & 0 & 0 \\ 0 & \# & 0 \\ 0 & 0 & \# \end{pmatrix}$ N×N
$\Rightarrow \langle a''|A|a'\rangle = \langle a'|A|a'\rangle\delta_{a',a''} = a'\delta_{a',a''}$ $|a_1\rangle\langle a_1| = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$
* in own basis, elements are diagonal i.e.
$\Rightarrow \hat{A} = \sum_{a'} a'|a'\rangle\langle a'| = \sum_{a'} a'\Lambda_{a'}$ $|a'\rangle\langle a'| = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & \vdots \\ 0 & & \ddots & \\ 0 & & & a_n \end{pmatrix}$ N

$\langle A\rangle = \langle\alpha|\hat{A}|\alpha\rangle = \sum_{a'}\sum_{a''}\langle\alpha|a''\rangle\langle a''|A|a'\rangle\langle a'|\alpha\rangle = \sum_{a'} a'|\langle a'|\alpha\rangle|^2$
$= \left(\sum_{a''} c_{a''}^*\langle a''|\right)A\left(\sum_{a'} c_{a'}|a'\rangle\right) = \sum_{a'a''} c_{a''}^* c_{a'}\langle a''|A|a''\rangle = \sum_{a'} a'|c_{a'}|^2$
• measured value a' • Probability of getting a' // Postulate probability

• When meas. is done, system goes into an eigenstate of the observable (ex $|a'\rangle$)
• Repeated measurements of same observable in succession yields same result
"average measured value"

$(\hat{B} = \mathbb{I}\hat{B}\mathbb{I} = \sum_{a',a''} |a'\rangle\langle a''|\underbrace{\langle a'|B|a''\rangle}_{= b'}$ & $\hat{B}$ operator represented w/ $\hat{A}$'s eigenkets as basis
• If $[A, B] = 0$ (compatible) $\Rightarrow$ if we measure $\hat{B}$ to get b', the following measurement of $\hat{A}$ always yields a' & get a' & subsequently
• $[A, B] = AB - BA$; $[S_i, S_j] = i\epsilon_{ijk}\hbar S_k$ $\epsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ cyclic} \\ -1 & \text{if } i,j,k \text{ or } 2,1,3 \\ 0 & \text{if } i=j, j=k \text{ or } i=k \end{cases}$
• $\{A, B\} = AB + BA$; $\{S_i, S_j\} = \frac{1}{2}\hbar^2\delta_{ij}$, $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$
• Probability is multiplicative $\rightarrow \boxed{A} \xrightarrow{|a'\rangle} \boxed{B} \xrightarrow{|b'\rangle} \boxed{C} \xrightarrow{|c'\rangle}$ $P = |\langle c'|b'\rangle|^2 |\langle b'|a'\rangle|^2$

change of basis/representation
2 orthonormal basis sets for 2 incompatible observables: $\{|a^i\rangle\}$, $i = 1, \dots N$ & $\{|b^i\rangle\}$, $i = 1, \dots N$
Then $\exists$ unitary operator $\hat{U}$ s.t. $|b^i\rangle = \hat{U}|a^i\rangle$ $\forall i$'s
$U|a^l\rangle = \left(\sum_k |b_k\rangle\langle a_k|\right)|a^l\rangle \rightarrow \langle a_k|a^l\rangle = \delta_{k,l} = \begin{cases} 1 & \text{if } k=l \\ 0 & \text{if } k\neq l \end{cases}$
so $k = l$, so $\sum_k |b^k\rangle\delta_{kl} = |b^l\rangle \Rightarrow U|a^l\rangle = |b^l\rangle$

Ex: Find unitary Transform. $\hat{U}$ s.t. $UBU^\dagger = D$ is diag.
$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ • $\alpha_1^\dagger B\alpha_1 = \lambda_1\alpha_1^\dagger\alpha_1 = \lambda_1$
we have: $U = (\alpha_1, \alpha_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ • $\alpha_2^\dagger B\alpha_1 = \lambda_1\alpha_2^\dagger\alpha_1 = 0$ • $\alpha_1^\dagger B\alpha_2 = \lambda_2\alpha_1^\dagger\alpha_2 = 0$
$U^\dagger = (\ )^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ • $\alpha_2^\dagger B\alpha_2 = \lambda_2\alpha_2^\dagger\alpha_2 = \lambda_2$
(trans. matrix)

Properties:
$|\alpha\rangle = \begin{pmatrix} a \\ b \\ c \\ \vdots \end{pmatrix}$ vector; $\langle\alpha| = (a^*\, b^* \dots)$
$|\alpha\rangle = \sum_a c_a|a\rangle = c_1|a_1\rangle + c_2|a_2\rangle + \dots$
$\langle\alpha| = \sum_a c_a^*\langle a| = c_1^*\langle a_1| + c_2^*\langle a_2| + \dots$
$\langle\alpha|\beta\rangle = (a^*\, b^*)\begin{pmatrix} c \\ d \end{pmatrix} = a^*c + b^*d$ inner product
$|\beta\rangle\langle\alpha| = \begin{pmatrix} c \\ d \end{pmatrix}(a^*\, b^*) = \begin{pmatrix} ca^* & cb^* \\ da^* & db^* \end{pmatrix}$ operator = matrix!
$\hat{X} = |\beta\rangle\langle\alpha|$; $\hat{X}^\dagger = |\alpha\rangle\langle\beta|$
$(|\beta\rangle\langle\alpha|)\cdot|\gamma\rangle = |\beta\rangle\cdot(\langle\alpha|\gamma\rangle)$

Postulates:
1) $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$
2) $\langle\alpha|\alpha\rangle \geq 0$
Normalized ket: $|\tilde{\alpha}\rangle = \left(\frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}\right)|\alpha\rangle$

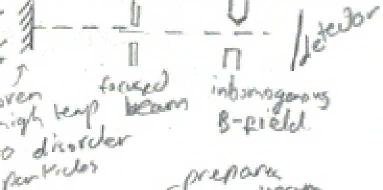
$\langle B|\hat{X}|\alpha\rangle = \langle B|(\hat{X}|\alpha\rangle)$
$= (\langle B|\hat{X})|\alpha\rangle$
Hermitian: $\langle B|\hat{X}|\alpha\rangle = \langle\alpha|\hat{X}|B\rangle^*$

$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle$, $= c_{a'}$
$= \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$ *
since $\langle a'|\alpha\rangle$ will collapse $\{|a''\rangle, |a'''\rangle \dots\}$ except $|a'\rangle$ ($\langle a'|a'\rangle$)

$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\,(|\alpha\rangle)$
$= 1$ closure
projection operator
"completeness Relation"
Identity operator (matrix)
$\langle\alpha|\alpha\rangle = \langle\alpha|\mathbb{I}|\alpha\rangle$
$= \langle\alpha|\left(\sum_{a'}|a'\rangle\langle a'|\right)|\alpha\rangle$
$= \sum_{a'}\langle\alpha|a'\rangle\langle a'|\alpha\rangle$
$= \sum_{a'}|\langle a'|\alpha\rangle|^2 = \sum_{a'}|c_{a'}|^2$

Uncertainty Rel^n:
$\Delta A \equiv A - \langle A\rangle$; $(\Delta A)^2$: "dispersion of A"
$\langle(\Delta A)^2\rangle = \langle(A^2 - 2A\langle A\rangle + \langle A\rangle^2)\rangle$
$= \langle A^2\rangle - \langle A\rangle^2$ "variance" or "mean sq deviation" "fuzziness"
$\langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}$
$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2$

Spin ½ system & Stern-Gerlach Experiment } classically expectation; Quantum (reality):



$H = -\vec{\mu}\cdot B$ magnetic moment

$F_z = \frac{\partial}{\partial z}(\vec{\mu}\cdot\vec{B}) = \mu_z \frac{\partial B}{\partial z}$

$\mu_z > 0$ ↓ force
$\mu_z < 0$ ↑ force

spin is quantized $+\frac{\hbar}{2}$ (spin ↑) $-\frac{\hbar}{2}$ (spin ↓)

$\left(\frac{N}{2}\right)\frac{1}{2} = \frac{N}{4}$ ← 50% chance

$\left(\frac{N}{4}\right)\left(\frac{1}{2}\right) = \frac{N}{8}$ ← 50% chance

spin-½: E.value eqⁿ: $S_z|S_z\pm\rangle = \pm\frac{\hbar}{2}|S_z\pm\rangle$ ("|+>" ket) (spin ½)

↓ matrix representations, operator

$\hat{S}_z = +\frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-| = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ *

$\left[\begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}\right] - \left[\begin{pmatrix}0\\1\end{pmatrix}(0\ 1) = \begin{pmatrix}0&0\\0&1\end{pmatrix}\right]$

$\hat{S}_+ = \hbar|+\rangle\langle-| = \hbar\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\hat{S}_- = \hbar|-\rangle\langle+| = \hbar\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$|S_x\rangle$ & $|S_y\rangle$ are mixtures of $|+\rangle$ & $|-\rangle$

$|S_x\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$; $|S_y\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$

General Dir. $\hat{n} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$

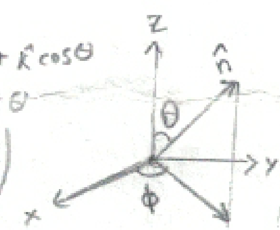
$\hat{S}_n = \vec{S}\cdot\hat{n} = S_x\sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta$

we know S_x, S_y, S_z so: $S_n = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$

S_n e.vals $\pm\frac{\hbar}{2}$, eig.vect.: $|+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$

$|\psi\rangle = |a||+\rangle + \sqrt{1-|a|^2}\,e^{i\phi}|-\rangle$

$|-\rangle_n = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}e^{i\phi}|-\rangle$



$0 \le \theta < \pi$
$0 \le \phi < 2\pi$

matrix representation of $\hat{S}_x, \hat{S}_y$ in $|\pm\rangle$ basis:

$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\hbar}{2}\left[(|+\rangle\langle-|) + (|-\rangle\langle+|)\right]$

$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\hbar}{2}\left[-i(|+\rangle\langle-|) + i(|-\rangle\langle+|)\right]$ } to find these:

let $S_x = \begin{pmatrix}a&b\\c&d\end{pmatrix}$ → E.val eqⁿ: $\begin{pmatrix}a&b\\c&d\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right] = \frac{\hbar}{2}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right]$

& $\begin{pmatrix}a&b\\c&d\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] = -\frac{\hbar}{2}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$ → $a+b = \hbar/2$ & $c+d = \hbar/2$; $a-b = -\hbar/2$ & $c-d = -\hbar/2$

→ $a=0,\ b=\hbar/2,\ c=\hbar/2,\ d=0$ ⇒ $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

Transformation Matrix: $\langle a^k|\hat{U}|a^l\rangle = \langle a^k|b^l\rangle$

$|\alpha\rangle = \sum_l c_l|a^l\rangle = \sum_l |a^l\rangle\langle a^l|\alpha\rangle$

$\langle b^k|\alpha\rangle = \sum_l \langle b^k|a^l\rangle\langle a^l|\alpha\rangle$

$= \sum_l \langle a^k|U^\dagger|a^l\rangle\langle a^l|\alpha\rangle$

$U = \sum_i |b^i\rangle\langle a^i|$

⇒ $U^\dagger = \sum_i |a^i\rangle\langle b^i|$

⇒ $\langle a^k|U^\dagger|a^l\rangle = \langle b^k|a^l\rangle$

⇒ $\langle b^k|\alpha\rangle = \sum_l \langle a^k|U^\dagger|a^l\rangle\langle a^l|\alpha\rangle$

i.e. (New) = $U^\dagger$(old)

How does operator transform from 1 basis to another? $X' = U^\dagger X U$

$\hat{X} = \sum_{i=1}^{3}|a^i\rangle\langle a^i|$ $\langle a^i|\hat{X}|a^j\rangle = \Pi\hat{X}\Pi$

$|a_1\rangle\langle a_1| = \begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0)$ $\xrightarrow{X_{ij}}$ $\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$

$|a_2\rangle\langle a_2| = \begin{pmatrix}0\\1\\0\end{pmatrix}(0\,1\,0) \longrightarrow \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$

$\longrightarrow \begin{pmatrix}X_{11}&X_{12}&X_{13}\\X_{21}&X_{22}&X_{23}\\X_{31}&X_{32}&X_{33}\end{pmatrix}$

sum of diagonal elements

Trace of an operator: $tr(\hat{X}) = \sum_{a'}\langle a'|X|a'\rangle$

Unitary Equivalent Observables: $A|a^l\rangle = a^l|a^l\rangle$, $U|a^l\rangle = |b^l\rangle$

⇒ $AI|a^l\rangle = a^l|a^l\rangle$ ⇒ $AU^{-1}U|a^l\rangle = a^l|a^l\rangle$

⇒ $UAU^{-1}(U|a^l\rangle) = a^l(U|a^l\rangle)$, $U|a^l\rangle = |b^l\rangle$

$UAU^{-1}|b^l\rangle = a^l|b^l\rangle$ i.e. states $|b^l\rangle$ are e.states of UAU^{-1} w/ same e.vals as A. ⇒ A & UAU^{-1} have same e.values — they are U.E. Obs

Properties list, trace, matrix work. look @ HW & do E1 ...

Matrix Representation of $\hat{A}$ in S_n - basis

$\hat{A} \rightarrow \hat{A}_n = \begin{pmatrix} \langle +n|\hat{A}|+n\rangle & \langle +n|\hat{A}|-n\rangle \\ \langle -n|\hat{A}|-n\rangle & \langle -n|\hat{A}|-n\rangle \end{pmatrix}$

$\hat{A} \rightarrow \hat{A}_n = S^{\dagger} A_z S$

w/ $S = \begin{pmatrix} \langle +z|+n\rangle & \langle +z|-n\rangle \\ \langle -z|+n\rangle & \langle -z|-n\rangle \end{pmatrix}$

& $A_z = \begin{pmatrix} \langle +z|\hat{A}|-z\rangle & \langle +z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|+z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix}$

Change of Basis (z to n) : States

$|\psi\rangle_z = (|+z\rangle\langle +z|)|\psi\rangle + (|-z\rangle\langle -z|)|\psi\rangle$

$= \left[|+n\rangle\langle +n| + |-n\rangle\langle -n|\right](|+z\rangle\langle +z|)|\psi\rangle$

$+ \left[|+n\rangle\langle +n| + |-n\rangle\langle -n|\right](|-z\rangle\langle -z|)|\psi\rangle$

$= \left(\langle +n|+z\rangle\langle +z|\psi\rangle + \langle +n|-z\rangle\langle -z|\psi\rangle\right)|+n\rangle$

$+ \left(\langle -n|+z\rangle\langle +z|\psi\rangle + \langle -n|-z\rangle\langle -z|\psi\rangle\right)|-n\rangle$

Matrix Representation $|\psi\rangle_n = \hat{S}^{\dagger}|\psi\rangle_z$:

$\begin{pmatrix} \langle +n|\psi\rangle \\ \langle -n|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle +n|+z\rangle & \langle +n|-z\rangle \\ \langle -n|+z\rangle & \langle -n|-z\rangle \end{pmatrix} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$

So for states : $|b'\rangle = \sum |a''\rangle\langle a''|b'\rangle$

→ this can be viewed as a matrix: $\hat{U}$

w/ $\hat{U}_i = \sum_m |b_m\rangle\langle a_m|$; i.e. $|a_i\rangle = \hat{U}_a|b_i\rangle$

b/c then $\hat{U}|a_k\rangle = \sum_m |b_m\rangle\langle a_m|a_k\rangle = |b_k\rangle$

Change of basis (z to n) : Operators

States are : $|\psi\rangle_n = \hat{S}^{\dagger}|\psi\rangle_z$ & ${}_n\langle\psi| = {}_z\langle\psi|S$

Relate $\langle\psi|\hat{A}|\psi\rangle$ in z basis to n basis:

${}_z\langle\psi|\hat{A}|\psi\rangle_z = {}_z\langle\psi|SS^{\dagger}A_z SS^{\dagger}|\psi\rangle_z$

$= {}_n\langle\psi|S^{\dagger}A_z S|\psi\rangle_n = {}_n\langle\psi|\hat{A}_n|\psi\rangle_n$

So, $\hat{A}_{new} = \hat{A}_n$ & $\hat{A}_{old} = \hat{A}_z \Rightarrow \hat{A}_{new} = S^{\dagger}\hat{A}_{old}S$

$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$|S_x \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$; $|S_y \pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle)$

$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$|1\rangle_x = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ → State found in $|1_y\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}$

Prob of finding each projection of ang. mom along z-axis?

$|0\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$P_{+\frac{1}{2}} = |\langle 1_z|1_y\rangle|^2$

$|-1\rangle_x = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

$P = |\langle\psi_{after}|\psi_{before}\rangle|^2$

$|1\rangle_x = \frac{1}{2}\left[|1\rangle + \sqrt{2}|0\rangle + |-1\rangle\right]$ ← in z-basis

$|1\rangle_z = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $|0\rangle_z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $|-1\rangle_z = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

• Initial state $|\psi\rangle_{old}$ can be represented in energy eigenbasis by using projection operator: $|\psi\rangle_{new} = \left(\sum_i |E_i\rangle\langle E_i|\right)|\psi\rangle_{old}$

So, $|\psi\rangle = c_1|a_1\rangle + c_2|a_2\rangle = |\psi\rangle_{old}$

given $\hat{H}$ in $\{|a_1\rangle, |a_2\rangle\}$ eigenbasis

$\det|H - \lambda I| = 0$ → $|E_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} a \\ b \end{pmatrix}$, $|E_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} c \\ d \end{pmatrix}$

⇒ $|E_1\rangle = \frac{1}{\sqrt{2}}(a|a_1\rangle + b|a_2\rangle)$

$|E_2\rangle = \frac{1}{\sqrt{2}}(c|a_1\rangle + d|a_2\rangle)$

Then $|\psi\rangle_{new\ basis} = (|E_1\rangle\langle E_1| + |E_2\rangle\langle E_2|)|\psi\rangle_{old}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$

unknown — $|+\rangle_x$ solve unknowns to get $\hat{S}_x$ in z-basis

not sure: for Hermitian $\hat{U} = U^{\dagger}$!

Completeness Relation: $\sum |u_i\rangle\langle u_i| = 1$

$D = U^{\dagger}CU$; U = Unitary Transformation: (will diagonalize the matrix & its elements will be its eigenvalues)

operator is always diagonal in its own basis

→ to measure probability it measuring in a new basis, must first convert the $|\psi\rangle$ the state is in to the new measured-basis

$|\psi_{in}\rangle = |1\rangle = \mathbb{I}|1\rangle = \left(|1\rangle_x{}_x\langle 1| + |0\rangle_x{}_x\langle 0| + |-1\rangle_x{}_x\langle -1|\right)|1\rangle$ (z basis)

$= \underbrace{{}_x\langle 1|1\rangle}_{a}|1\rangle_x + \underbrace{{}_x\langle 0|1\rangle}_{b}|0\rangle_x + \underbrace{{}_x\langle -1|1\rangle}_{c}|-1\rangle_x$

$P_{+1} = |{}_x\langle 1|\psi_{in}\rangle|^2 = |C_n|^2$

i.e. you can't find C_n coefficients $e^{i\theta}$, $e^{i\varphi}$ include

$|\psi\rangle = a|+\rangle + b|-\rangle$, a measurement prob of results of w/ spin component S_x

→ $|{}_x\langle +|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(\langle +| + \langle -|)\right||+\rangle|^2$

$|{}_x\langle -|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left[\langle +| - \langle -|\right]|-\rangle\right|^2$

• General 2×2 spin-$\frac{1}{2}$: $|\pm\rangle_n = \cos\frac{\theta}{2}|+\rangle \pm ie^{i\varphi}\sin\frac{\theta}{2}|-\rangle$ (z-basis)

→ Normalized e.values & eigenvectors of L_x in L_z basis

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be eigenstate — $a|1\rangle + b|0\rangle + c|-1\rangle = |\psi\rangle$

$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \det|L_x - I\lambda| = 0 \Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm 1$

L_x, L_z: $|\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for $\lambda_1 = 0$, $|\lambda_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ for $\lambda_2 = 1$, $|\lambda_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$

Particle is in $L_z = -1$, L_x measured: $P = |\langle L_x|L_{z=-1}\rangle|^2$ for λ_3

→ L_z^2 is measured gives +1 result, what is state after measurement?

So $L_z^2|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \lambda\begin{pmatrix} 1/2 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \Rightarrow \begin{matrix} 1/2 = \lambda\, 1/2 \\ 1/\sqrt{2} = \lambda\, 1/\sqrt{2} \end{matrix}$ 2 cases when $\lambda = +1$

So $\hat{P}_{+1} = |1\rangle\langle 1| + |-1\rangle\langle -1|$

Now $|\psi_{after}\rangle = \dfrac{\hat{P}_{+1}|\psi\rangle_{old}}{\sqrt{\langle\psi|\hat{P}_{+1}|\psi\rangle}} = (\ )$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

→ Given: $H = \Delta(|L\rangle\langle R| + |R\rangle\langle L|) = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$ → $\lambda = \pm\Delta$

apply time evolution: $|\alpha, t\rangle = U|\alpha\rangle = e^{-iHt/\hbar}$ $|R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle = (|R\rangle\langle R| + |L\rangle\langle L|)|\alpha\rangle$

Probability on left: $P = |\langle L|\alpha, t\rangle|^2$

Rewrite in term of energy basis:

$|E_1\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$

$|E_2\rangle = \frac{1}{\sqrt{2}}(|R\rangle - |L\rangle)$

$|L\rangle = \sqrt{2}|E_1\rangle - |R\rangle$

& $|R\rangle = \sqrt{2}|E_2\rangle + |L\rangle$ Plug in

⇒ $|R\rangle = \frac{\sqrt{2}}{2}(|E_1\rangle + |E_2\rangle)$

$|L\rangle = \frac{\sqrt{2}}{2}(|E_1\rangle - |E_2\rangle)$

→ Given: $P_+ = \frac{1}{2}$ $P_{+x} = \frac{3}{4}$ $P_{+y} = 0.067$ $P_{-y} = 0.93$ Find $|\psi_n\rangle$

$|\psi_{in}\rangle = a|+\rangle + b|-\rangle$; $P_+ = |\langle +|\psi\rangle|^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}e^{i\varphi}$

$P_{+x} = \frac{3}{4} = |{}_x\langle +|\psi\rangle|^2 = \left|\left[\frac{1}{\sqrt{2}}(\langle +| + \langle -|)\right]\left[a|+\rangle + b|-\rangle\right]\right|^2 = \left|\frac{1}{2} + \frac{1}{2}e^{i\varphi}\right|^2 = \frac{1}{2}(1 + \cos\varphi)$

$\varphi = \frac{\pi}{3}$

$\hat{A}$ has 2 normalized eigenstates $|\psi_1\rangle, |\psi_2\rangle$ w/ e.values a_1, a_2

$\hat{B}$ has " " " $|\phi_1\rangle, |\phi_2\rangle$ w/ e.values b_1, b_2

w/ $|\psi_1\rangle = \frac{1}{\sqrt{10}}(|\phi_1\rangle + 3|\phi_2\rangle)$ & $|\psi_2\rangle = \frac{1}{\sqrt{10}}(3|\phi_1\rangle - |\phi_2\rangle)$

$\hat{A}$ measured → a_1. If B is measured, then $\hat{A}$ again, what probability of getting a_1 again? Probability to get a_2 instead?

Soln: a_1 means collapsed to $|\psi_1\rangle$

B measurement → $|\langle\phi_1|\psi_1\rangle|^2 = P_{b_1}$, $|\langle\phi_2|\psi_1\rangle|^2 = P_{b_2}$ — $|\psi\rangle$ state collapses to one of these

A measurement again → if B measurement gave $|\phi_1\rangle$ (i.e. b_1): $|\langle\psi_1|\phi_1\rangle|^2 = P_{a_1}$ ①

↓ if B measured gave $|\phi_2\rangle$ (i.e. b_2): $|\langle\psi_1|\phi_2\rangle|^2 = P_{a_1}$ ②

(Prob of getting a_1 i.e. $|\psi_1\rangle$)

⇒ total probability of a_1! ① × ② $= \frac{82}{100}$

Properties:

$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$

- $\langle\psi|(\alpha|\phi\rangle+\beta|\omega\rangle) = \alpha\langle\psi|\phi\rangle + \beta\langle\psi|\omega\rangle$
- $(\langle\alpha\psi| + \langle\beta\omega|)|\phi\rangle = \alpha^*\langle\psi|\phi\rangle + \beta^*\langle\psi|\omega\rangle$
- $\langle\psi|\psi\rangle \geq 0$

Other Remember:

$z + z^* = 2\,\mathrm{Re}(z)$

$(|\psi\rangle\langle\phi|)|\chi\rangle = |\psi\rangle(\langle\phi|\chi\rangle)$; $\hat{M}$, $\hat{P}_i$, $\in \mathbb{C}$

$\mathbb{I} = \sum_{i=1}^{n} |u_i\rangle\langle u_i| = \sum \hat{P}_i$

$T_{ij} = \langle u_i|\hat{T}|u_j\rangle$ (row, column)

$\hat{A} = a|0\rangle\langle 0| + b|0\rangle\langle 1| \ldots = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- $\mathrm{Tr}(\hat{T}) = \sum_{i=1}^{n} T_{ii}$
- $\langle\omega|\hat{T}|u\rangle = \langle\omega|\hat{T}v\rangle = \langle v|\omega\rangle^*$
- $H^\dagger = [H^T]^*$; Hermitian: $H = H^\dagger$
- $\lambda\langle u|(\hat{A}\hat{B})|v\rangle \Rightarrow \lambda^*\langle v|B^\dagger A^\dagger|u\rangle$
  - To find adjoint of general expression
- Hermitian $\Rightarrow$ $\langle\omega|\hat{T}|u\rangle = \langle u|T|\omega\rangle^*$
- Show Hermitian operators have e.val $\in \mathbb{R}$:
  $\langle a|(A|a\rangle) = \langle a|(A|a\rangle) = \lambda\langle a|a\rangle$
  $(\langle a|A)|a\rangle = \lambda^*\langle a|a\rangle$
  subtract $\Rightarrow$ $0 = (\lambda - \lambda^*)\langle a|a\rangle$
- Show e.kets w/ different e.val are orthonormal:
  - let $A|\lambda_1\rangle = a_1|\lambda_1\rangle$ & $A|\lambda_2\rangle = a_2|\lambda_2\rangle$ — ②
  $\Rightarrow \langle\lambda_1|A = a_1\langle\lambda_1|$ — ①
  multiply ① by $|\lambda_2\rangle$ & ② by $\langle\lambda_1|$ B.S.
  subtract: $0 = (a_1 - a_2)\langle\lambda_1|\lambda_2\rangle$
  since $a_1 \neq a_2 \Rightarrow \langle\lambda_1|\lambda_2\rangle = 0$
- if $[A,B] = 0 \Rightarrow$ they share common eigenstates:
  let $A|a\rangle = a|a\rangle \Rightarrow \hat{B}\hat{A}|a\rangle = B(a|a\rangle)$
  Since $[A,B] = 0 \Rightarrow \hat{A}(\hat{B}|a\rangle) = a(B|a\rangle)$
  So $B|a\rangle$ is also e.state of $\hat{A}$ w/ e.value a.
  Since $\nexists$ no degeneracy, each e.value is unique eigenstate so $B|a\rangle$ must be some scalar multiple of $|a\rangle$ — i.e. $\hat{A}(B|a\rangle) = \lambda|a\rangle = a(b|a\rangle)$
  so $|a\rangle$ is also e.state of B. (i.e. $B|a\rangle = b|a\rangle$.)
- Unitary Matrices: $UU^\dagger = U^\dagger U = \mathbb{I}$ $\Rightarrow U^\dagger = U^{-1}$
  - its eigenvalues have unit magnitude $|a_n|^2 = 1$
  - its rows & columns form orthonormal set
  - $(\mathrm{row})_n \cdot (\mathrm{row})_m$ must $= 0$ (show orthonormal)
- Anti-Hermitian: $A^\dagger = A^{-1}$
- $[A,B] = AB - BA$
- $[A,B] = -[B,A]$
- $[A+B,C] = [A,C] + [B,C]$
- $[x,p] = i\hbar$; $[x,x] = [p,p] = 0$
- $[A,BC] = [A,B]C + B[A,C]$
- $\rightarrow$ Complete set of commut. observables: $[A,B] = [A,C] = [B,C] = \ldots = 0$ i.e. $\exists$ common eigenvectors (unique to a constant).

Cauchy Schwartz:

$|\langle\phi|\psi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle$

Proof: $|f\rangle = |\phi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}|\psi\rangle$; $\langle f|f\rangle \geq 0$ $\Rightarrow$ use: $\langle\psi|\phi\rangle^*\langle\phi|\psi\rangle = \langle\phi|\psi\rangle\langle\phi|\psi\rangle = |\langle\phi|\psi\rangle|^2$

Triangle Inequality: $\sqrt{\langle\psi+\phi|\psi+\phi\rangle} \leq \sqrt{\langle\psi|\psi\rangle} + \sqrt{\langle\phi|\phi\rangle}$

Proof: $\mathrm{Re}(|\langle\psi|\phi\rangle|) \leq |\langle\psi|\phi\rangle|$; expand.

note: $\langle\psi|\phi\rangle + \langle\phi|\psi\rangle = \langle\psi|\phi\rangle + \langle\psi|\phi\rangle^* = 2\,\mathrm{Re}(\langle\psi|\phi\rangle)$

i.e. if $\mathbb{R}$: $\langle\psi|\phi\rangle^* = \langle\psi|\phi\rangle$.

use Cauchy Schwartz

- $\hat{P} = \hat{P}^\dagger$, $\hat{P} = \hat{P}^2$ (projection operators)
- $(\Delta x)(\Delta p) \geq \frac{\hbar}{2}$; $(\Delta A)^2 = \langle(\hat{A} - \langle A\rangle)^2\rangle$
- $(\Delta A)(\Delta B) \geq \frac{[A,B]}{2i}$; $\Delta x\,\Delta p \geq \frac{\hbar}{2}$
- $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$; $P = |\langle\psi_{new}|\psi_{old}\rangle|^2 = \left|\int_{-\infty}^{\infty}\psi_{new}^*(x)\,\psi_{old}(x)\,dx\right|^2$
- $\psi(x) = \langle x|\psi\rangle = \langle x|\mathbb{I}|\psi\rangle = \langle x|\int|p\rangle\langle p|\,dp\,|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int e^{ipx/\hbar}\,\tilde{\psi}(p)\,dp$.
- $|\psi\rangle = \int c(a)\,|a\rangle\,da$
- $\langle\phi|\psi\rangle = \sum b_i^* c_i \Rightarrow \int b(a)^* c(a)\,da$
- $\langle\psi|A|\psi\rangle = \int_{-\infty}^{\infty}\psi(x)^* A\,\psi(x)\,dx$
- $\langle\phi|\psi\rangle = \int\phi^*\psi\,dx$
- $|\psi\rangle = \sum_j \psi(x)|x\rangle$
- $P_{\lambda_i} = |\langle a_i|\psi\rangle|^2 = \langle a_i|\psi\rangle(\langle a_i|\psi\rangle^*) = \langle a_i|\psi\rangle\langle\psi|a_i\rangle = \langle\psi|a_i\rangle\langle a_i|\psi\rangle = \langle\psi|\hat{P}_i|\psi\rangle$
- $\hat{A} = \sum_{ij} A_{ij}|u_i\rangle\langle u_j|$
- $\hat{A} = \sum\lambda_i|a_i\rangle\langle a_i| = \sum\lambda_i\hat{P}_i$ — Spectral Decomposition: operator written in terms of projection operators
- State immediately after measurement: $|\psi\rangle \rightarrow \left(\frac{1}{\sqrt{\langle\psi|\hat{P}_j|\psi\rangle}}\right)\hat{P}_j|\psi\rangle$
- $\langle H\rangle = \sum[P(E_i)]E_i = \langle\psi|\hat{H}|\psi\rangle$ (prob., e.values)
- Time evolution:
  $i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi(t_0)\rangle$
  $|\psi(t)\rangle = e^{-iE_i\Delta t/\hbar}|\psi(t_0)\rangle$
- Product of 2 Unitary operators is unitary:
  Pf) $(AB)(AB)^\dagger = (AB)(B^\dagger A^\dagger)$
  since $(AB)^\dagger = B^\dagger A^\dagger$ ← for Hermitian $AB = B^\dagger A^\dagger = BA$
  $= A(BB^\dagger)A^\dagger = AIA^\dagger = AA^\dagger = I$ $\Rightarrow$ so AB is unitary

operators

- $\hat{T}(\alpha|u\rangle + \beta|v\rangle) = \alpha\hat{T}|u\rangle + \beta\hat{T}|v\rangle$
- $(\hat{T}+\hat{S})|u\rangle = \hat{T}|u\rangle + \hat{S}|u\rangle$
- $(\hat{T}\hat{S})|u\rangle = \hat{T}(\hat{S}|u\rangle)$.

$|\psi\rangle = \sum_i c_i|u_i\rangle$; $c_1, c_2 \ldots$; $|\psi\rangle = \begin{pmatrix}\langle u_1|\psi\rangle \\ \langle u_2|\psi\rangle \\ \vdots \\ \langle u_n|\psi\rangle\end{pmatrix}$

given $|a\rangle, |b\rangle, |c\rangle$ linearly independent?

$a_1|a\rangle + a_2|b\rangle + a_3|c\rangle = 0$

$(\ ) + (\ ) + (\ ) = (\ ) = 0$

$\hookrightarrow$ if can only be true iff $a_1 = a_2 = a_3 = 0$

$\Rightarrow$ yes, linearly independent

Gram-Schmidt Procedure

- If you have NOT orthonormal basis set $\{|u_i\rangle\}$ $\Rightarrow$ to build one:
  set $|\omega_1\rangle = |u_1\rangle$
  $\Rightarrow |\omega_2\rangle = |u_2\rangle - \frac{\langle\omega_1|u_2\rangle}{\langle\omega_1|\omega_1\rangle}|\omega_1\rangle$
  $|\omega_3\rangle = |u_3\rangle - \frac{\langle\omega_1|u_3\rangle}{\langle\omega_1|\omega_1\rangle}|\omega_1\rangle - \frac{\langle\omega_2|u_3\rangle}{\langle\omega_2|\omega_2\rangle}|\omega_2\rangle$
  ⋮
  Then orthonormal set will be $\{|v_i\rangle\}$
  so, $|v_1\rangle = \frac{|\omega_1\rangle}{\sqrt{\langle\omega_1|\omega_1\rangle}}$; $|v_2\rangle = \frac{|\omega_2\rangle}{\sqrt{\langle\omega_2|\omega_2\rangle}}$, $|v_3\rangle = \frac{|\omega_3\rangle}{\sqrt{\langle\omega_3|\omega_3\rangle}}$

$\rightarrow$ Given $\hat{H}|u_1\rangle = E|u_1\rangle$ $H|u_2\rangle = E|u_2\rangle$
$H|u_3\rangle = 2E|u_3\rangle$ $H|u_4\rangle = \ldots$
$|\psi\rangle = a|u_1\rangle + b|u_2\rangle + c|u_3\rangle + \ldots$
Prob of getting eigenvalue E:
$\hat{P}_E = \sum|u_i\rangle\langle u_i| = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|$ ← projection operator
$|\psi_{after}\rangle = \frac{\hat{P}_E|\psi\rangle}{\sqrt{\langle\psi|\hat{P}_E|\psi\rangle}}$
(E is in both state $|u_1\rangle$ & $|u_2\rangle$)

L_z^2 is measured

prob of finding between 0 & L: $P = \int_0^L\psi^*(x)\,\psi(x)\,dx$

- $p = -i\hbar\frac{\partial}{\partial x}$: $\langle p\rangle = \int_{-\infty}^{\infty}\psi^*(x)\,\hat{p}\,\psi(x)\,dx$
- $\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta)$
- $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin(\theta)$
- $e^{iA} = \sum_{j=0}^{\infty}\frac{(iA)^j}{j!}$

$\langle A\rangle = \langle B|A|B\rangle$; let $e^{i\theta}|c\rangle = e^{i\theta}\sum_i a_i|i\rangle = |B\rangle$

$\Rightarrow \langle A\rangle = (\sum e^{-i\theta}a_i^*\langle i|)A(\sum e^{i\theta}a_i|i\rangle) = \cancel{e^{-i\theta}e^{i\theta}}\langle c|A|c\rangle$ ← phase factor doesn't matter

- $\hat{X}|\psi\rangle = x_i|\psi\rangle \rightarrow \langle\psi|X^\dagger = \langle\psi|x_i^*$
  $\Rightarrow \langle\psi|X^\dagger X|\psi\rangle = \langle\psi|x_i^* x_i|\psi\rangle = |c|^2\langle\psi|\psi\rangle = \mathrm{Prob}$; c_i
- $|\psi\rangle = (\sum_i|a_i\rangle\langle a_i|)|\psi\rangle = \sum_i(\langle a_i|\psi\rangle)|a_i\rangle$
- $\hat{p}\hat{x} \neq \hat{x}\hat{p}$.

$\hat{H} = \hbar\omega\begin{pmatrix}1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 5/2\end{pmatrix}$

$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$: $\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ — square well

$E_n = (n + 1/2)\hbar\omega$ for H. Osc.

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$; $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$

P4

$$|\psi_{in}\rangle = |11\rangle = |1\rangle = \underbrace{\left(|1\rangle_x\,{}_x\langle 1| + |0\rangle_x\,{}_x\langle 0| + |-1\rangle_x\,{}_x\langle -1|\right)}_{=\mathbb{I}}|1\rangle$$

$$= \underbrace{{}_x\langle 1|1\rangle}_{=a}|1\rangle_x + \underbrace{{}_x\langle 0|1\rangle}_{=b}|0\rangle_x + \underbrace{{}_x\langle -1|1\rangle}_{=c}|-1\rangle_x$$

set 1

we keep this in Sx-basis

$$P_{+\hbar} = |{}_x\langle 1|\psi_{in}\rangle|^2 = \frac{1}{4}$$

$$= \left|{}_x\langle 1|\left(({}_x\langle 1|1\rangle)|1\rangle_x + ({}_x\langle 0|1\rangle)|0\rangle_x + ({}_x\langle -1|1\rangle)|-1\rangle\right)\right|^2$$

$$= |\underbrace{({}_x\langle 1|1\rangle)}_{=a}\,\cancel{{}_x\langle 1|1\rangle_x}^{1}|^2 = \frac{1}{4}$$

$$= |a|^2 = \frac{1}{4} \Rightarrow a = \frac{1}{2}$$

$$P_{0\hbar} = |{}_x\langle 0|\psi_{in}\rangle|^2 = \frac{1}{2}$$

$$= |\underbrace{({}_x\langle 0|1\rangle)}\,\cancel{{}_x\langle 0|0\rangle_x}|^2$$

$$= |b|^2 = \frac{1}{2} \Rightarrow b = \frac{1}{\sqrt{2}}e^{i\alpha}$$

$$P_{-\hbar} = |{}_x\langle -1|\psi_{in}\rangle|^2 = \frac{1}{4}$$

$$= |\underbrace{{}_x\langle -1|1\rangle}\,{}_x\langle -1|-1\rangle_x|^2$$

$$|c|^2 = \frac{1}{4} \Rightarrow c = \frac{1}{2}e^{i\beta}$$

$$|\psi_{in}\rangle = \frac{1}{2}|1\rangle_x + \frac{1}{\sqrt{2}}e^{i\alpha}|0\rangle_x + \frac{1}{2}e^{i\beta}|-1\rangle_x$$ for $|S_z=0\rangle$

set 2

$$|S_z=0\rangle \longrightarrow |\psi_{in}\rangle = \left(|1\rangle_x\,{}_x\langle 1| + |0\rangle_x\,{}_x\langle 0| + |-1\rangle_x\,{}_x\langle -1|\right)|0\rangle$$

$$= ({}_x\langle 1|0\rangle)|1\rangle_x + ({}_x\langle 0|0\rangle)|0\rangle_x + ({}_x\langle -1|0\rangle)|-1\rangle_x$$

$$P_{\hbar} = |{}_x\langle 1|\psi_{in}\rangle|^2 = \frac{1}{2}$$

$$= |\underbrace{({}_x\langle 1|0\rangle)}_{e}\,\cancel{{}_x\langle 1|1\rangle_x}^{1}|^2 = |e|^2 = \frac{1}{2}$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

~~measured probab~~ $\quad |{}_x\langle 0|\psi_{in}\rangle|^2$

$$P_{0x} = |\underbrace{({}_x\langle 0|0\rangle)}_{f}\ {}_x\langle 0|0\rangle_x|^2 = |f|^2 = 0 \longrightarrow f = 0$$

$$P_{-1x} = |\underbrace{({}_x\langle -1|0\rangle)}_{g}\ {}_x\langle -1|-1\rangle_x|^2 = |g|^2 = \tfrac{1}{2} \longrightarrow g = \frac{1}{\sqrt{2}}$$

sol 3 $\quad |S_z = -1\rangle$

$$\psi_{in} = |-1\rangle = \underbrace{\left(|1\rangle_x\, {}_x\langle 1| + |0\rangle_x\, {}_x\langle 0| + |-1\rangle_x\, {}_x\langle -1|\right)}_{=\mathbb{I}} |-1\rangle$$

$$P_{1x} = |{}_x\langle 1|\psi_{in}\rangle|^2 = \left|{}_x\langle 1|\left(({}_x\langle 1|-1\rangle)|1\rangle_x + ({}_x\langle 0|-1\rangle)|0\rangle_x + ({}_x\langle -1|-1\rangle)|-1\rangle_x\right)\right|^2$$

$$= |({}_x\langle 1|-1\rangle)\ {}_x\langle 1|1\rangle_x|^2 = \tfrac{1}{4}$$

$$= |h|^2 = \tfrac{1}{4} \longrightarrow h = \frac{1}{2}$$

$$P_{0x} = |\underbrace{{}_x\langle 0|-1\rangle}_{i}\ {}_x\langle 0|0\rangle_x|^2 = \tfrac{1}{2} \qquad |i|^2 = \tfrac{1}{2} \longrightarrow i = \frac{1}{\sqrt{2}}$$

$$P_{-1x} = |\underbrace{{}_x\langle -1|-1\rangle}_{j}\ {}_x\langle -1|-1\rangle_x|^2 = \tfrac{1}{4} \longrightarrow |j|^2 = \tfrac{1}{4} \longrightarrow j = \frac{1}{2}$$

$$|\psi_{in}\rangle = \frac{1}{2}|1\rangle_x + \frac{1}{\sqrt{2}} e^{i\theta}|0\rangle_x + \frac{1}{2} e^{i\phi}|1\rangle_x$$

for $|S_z = 0\rangle$

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}|1\rangle_x + 0|0\rangle_x + \frac{1}{\sqrt{2}} e^{i\delta}|1\rangle$$

~~Soln Summary~~

→ Identify matrix of S_x: $|1\rangle_x\, {}_x\langle 1| + |0\rangle_x\, {}_x\langle 0| + |-1\rangle_x\, {}_x\langle -1|$ to that input state $|S_z = 0\rangle$ in S_x basis ex

→ plug the against $|{}_x\langle 1|\psi_{in}\rangle|^2$ to solve for coefficients given probabilities