Hartwick College

Senior Thesis

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Pre-Thesis:

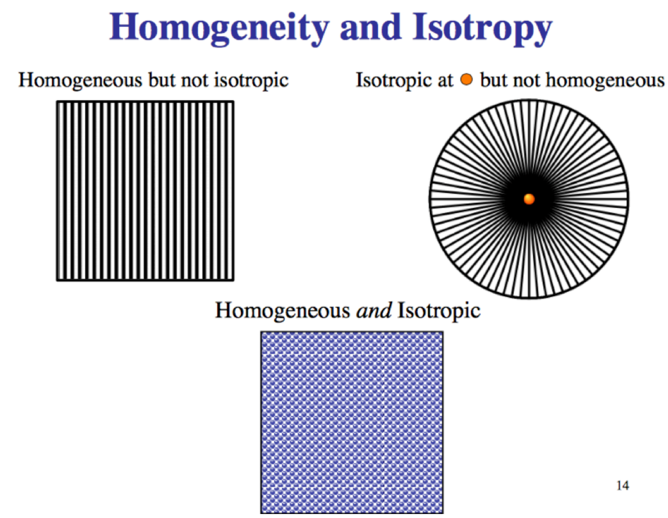
An Introduction to Cosmology and General Relativity

Brief Summary of Cosmology

Cosmology begins with two primary assumptions:

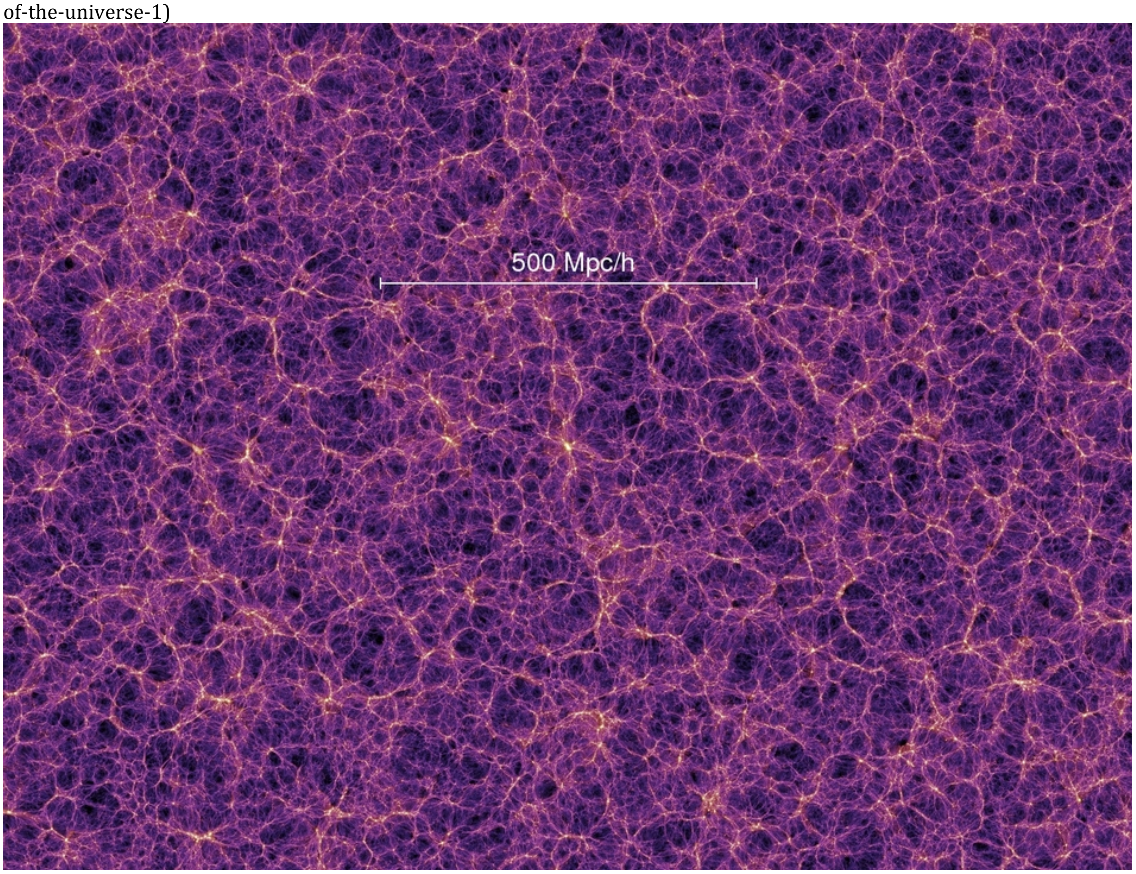
1. The universe is *homogenous*—the same everywhere
2. The universe is *isotropic*—the same in every direction

This is called ***the Cosmological Principle***. (The two do not imply one another).
In other words, we are not in any special place in the universe. Note that this is applicable to the universe on the grand scale, and therefore large-scale approximations hold well for our observations. It is on a scale larger than 250 million light years (https://en.wikipedia.org/wiki/Cosmological_principle), or sometimes considered approximately 100 Mega-parsec (1 parsec = 3.26 light years). On the smaller scale, galaxy clusters do look different because matter is not uniformly distributed.



https://www.astro.caltech.edu/~george/ay21/Ay21_Lec02.pdf

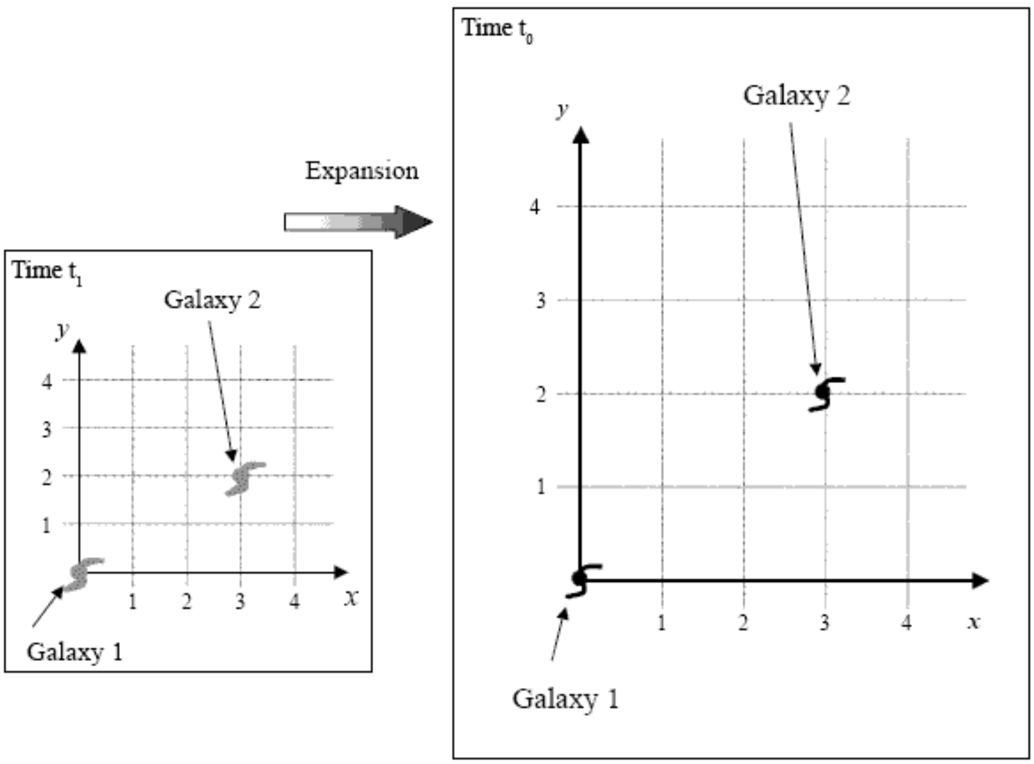
The Universe on a large scale (http://www.sun.org/images/structure-of-the-universe-1)



Hubble's Constant is a proportionality parameter. It is believed to be a constant in space but is time variant as the expansion rate of the universe changes.
It is given as: $H(t) = v/r$ where v is the velocity and r is the distance. The current time is the age of the universe at t_o = 13.7 billion years. The Hubble constant is commonly written as $H_o = 100h$ in which the measured value of today is approximately .72 plus/minus .08 (it changes as the universe has a varying expansion rate).

Scale factor measures the universe's expansion rate. It shows us how physical distances are growing in time.

It is defined given by $\vec{r} = a(t)\vec{x}$ where $\vec{x}$ is the *comoving distance* and $\vec{r}$ is the *physical distance.* In other words, the comoving distance is the assigned (fixed) coordinate of the galaxy, and as space expands (represented by the scale factor) the comoving distance stay the same but the physical distance increase. (See image below).



(https://lonewolfonline.net/cosmic-scale-factor/)

Now we see:

$$H(t) = \frac{v}{r} = \frac{r\prime}{r} = \frac{[a(t)x]\prime}{a(t)x} = \frac{\dot{a}(t)}{a(t)} \qquad (\dot{a}(t) \text{ means } \frac{da}{dt})$$

Arguably the most important equation in cosmology is the **Friedmann Equation:**

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda c}{3}$$

Note: *c = 1 (speed of light is a constant commonly set to 1); a_o = 1 (the scale factor of today); k = -1,0, +1; G is the gravitational constant 6.67*10^{-11} m^3 kg^{-1} s^{-2}*

ρ represents the mass density, while k represents the curvature of the universe with three possible geometries: "Open", "flat" or "closed" which follows hyperbolic, Euclidean or spherical geometry, respectively. Also Λ represents the quantity, which seems to cause the universe to accelerate in its expansion—the so-called *dark energy* is associated with this term. Overall it describes the expansion of the universe.

The Fluid Equation looks at the ρ value in the Friedmann Equation. This value represents the *density of material* in the universe, and how it evolves with time. It is given below:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

Note: $\dot{\rho} = \frac{d\rho}{dt}$; P is the pressure of the material

The Acceleration Equation describes the acceleration of the scale factor. It is given as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}$$

Observations

Given the Friedmann Equation—$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda c}{3}$—if we set $k = \Lambda = 0$ we can rearrange the equation solving for rho. This is called the **critical density:**

$$\rho_c(\mathrm{t}) = \frac{3H^2}{8\pi G}$$

The Density Parameter is a dimensionless quantity, which is useful for specifying the density of the universe. Given by:

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)}$$

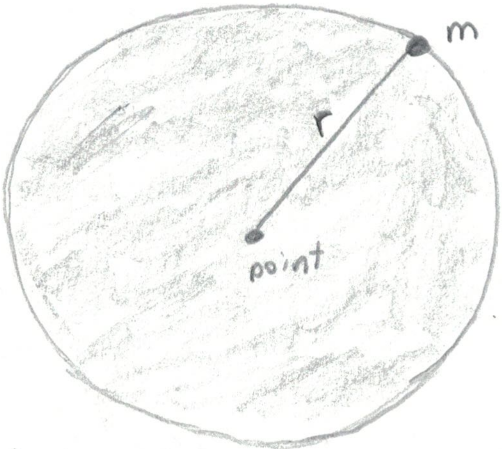
Where $\rho_c(t)$ is the amount of energy required for flatness of a universe while the numerator is the amount of energy in the universe. This relationship is observed today to be 1.005 ± 0.007 indicating an almost-flat universe.

Friedmann Equation Derivation — Newtonian Method

Suppose we have a uniform expanding medium with mass density = $\frac{\text{mass}}{\text{Volume}}$ = ρ.

Since the Cosmological Principle says that universe is the same everywhere, consider any point to be its center.

Consider a particle distance r away with mass m.



Newton's Theorem

In spherically symmetric distribution of matter, a particle feels no force at greater radii.

Furthermore, material at smaller radii gives exactly the force which one would experience if All material was concentrated at the central point.

By Newton's Theorem, the particle only feels force from material at smaller radii.

Material has: $\rho = \frac{M}{V} \Rightarrow M = \rho V = \left(\frac{4}{3}\pi r^3\right)\rho$

so Law of Gravitation: $F = \frac{GMm}{r^2} = \frac{G(\frac{4}{3}\pi r^3 \rho)m}{r^2} = \frac{4\pi}{3} G r m \rho$

Gravitational Potential Energy: $V = -\frac{GMm}{r} = -\frac{G(\frac{4}{3}\pi r^3 \rho)m}{r} = -\frac{4\pi}{3} G r^2 m \rho$

Kinetic Energy of particle: $T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{dr}{dt}\right) = \frac{1}{2} m \dot{r}^2$ ($\dot{r}$ = derivative of position over time)

Conservation of Energy: $E_{TOTAL} = U = T + V = \frac{1}{2} m \dot{r}^2 + \left(-\frac{4\pi G \rho m r^2}{3}\right)$

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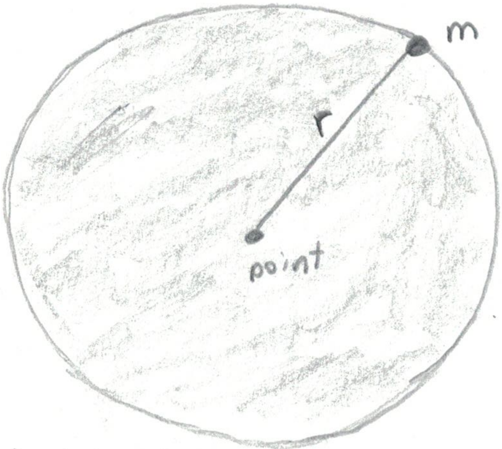

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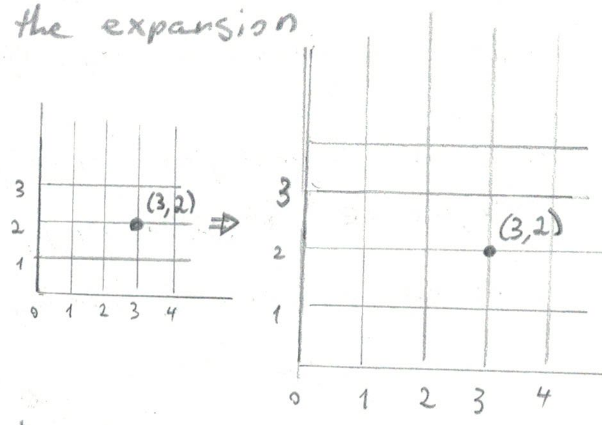
⇒ Now we introduce comoving coordinates: a coordinate system in which coordinates are carried along in the expansion.

We are allowed to do this since the Universe is homogeneous.



Since Expansion is uniform, we have:

$\vec{r} = a(t)\,\vec{x}$ $\vec{r}$ = physical distance

$\vec{x}$ = comoving distance

$a(t)$ = scale factor

* above: coordinate system is the same but physical distance has increased

so, $\vec{x}$ are the labels attached to the galaxies at all times while $\vec{r}$, the physical distance, gets bigger in time since the whole coordinate grid expands.

Using this we rewrite Energy conservation:

$$U = \frac{1}{2}m\left[(a(t)\vec{x})^2\right]' - \frac{4\pi G\rho m}{3}\left[a(t)\vec{x}\right]$$

$$= \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi G\rho m a^2x^2}{3}$$

* $\dot{r} = \frac{d\vec{r}}{dt} = \frac{d(a(t)\vec{x})}{dt} = \vec{x}\,\underbrace{\frac{d(a(t))}{dt}}_{=\,\dot{a}}$

only scale factor changes in time, $\vec{x}$ is time-independent!

multiply both sides by $\frac{2}{ma^2x^2}$ (to isolate a-terms):

$$\frac{2U}{ma^2x^2} = \frac{2}{ma^2x^2}\left(\frac{1}{2}m\dot{a}^2x^2\right) - \frac{2}{ma^2x^2}\left(\frac{4\pi G\rho m a^2x^2}{3}\right)$$

$$\frac{2U}{mx^2a^2} = \frac{\dot{a}^2}{a^2} - \frac{8\pi G\rho}{3}$$

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \left(\frac{2U}{mx^2}\right)\frac{1}{a^2}$$

Set $Kc^2 = \frac{-2U}{mx^2}$

*K is independent of x and of time (since all other terms on right side are)

∴ K = constant

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}}$$

Fluid Equation Derivation

1st Law of Thermodynamics: Conservation of Energy

$$\Rightarrow \underbrace{dE}_{\text{energy}} + \underbrace{P}_{\text{pressure}} \underbrace{dV}_{\text{volume}} = \underbrace{T}_{\text{temperature}} \underbrace{dS}_{\text{entropy}}$$

mass of ball : $m = \underbrace{\left(\frac{4}{3}\pi r^3\right)}_{\text{volume}} \rho$ density = $\frac{\text{mass}}{\text{Volume}}$

$E = mc^2 = \left(\frac{4}{3}\pi r^3 \rho\right) c^2$; now $r = a(t)$ <scale factor>

(i) $\frac{dE}{dt} = \frac{4}{3}\pi c^2 \rho \, 3(a(t))^2 \frac{da}{dt} + \frac{4}{3}\pi c^2 [a(t)]^3 \frac{d\rho}{dt}$ <chain rule>

(ii) $\frac{dV}{dt} = 4\pi a(t)^2 \frac{da}{dt}$ <rate of volume change>

using (i) get rid of dt: $dE = \frac{4}{3}\pi c^2 \rho \, 3a^2 \dot{a} + \frac{4}{3}\pi c^2 a^3 \dot{\rho}$

$= \frac{4}{3}\pi c^2 \left(3a^2\dot{a}\rho + a^3\dot{\rho}\right)$

using (ii), get rid of dt:

$dV = 4\pi a^2 \dot{a}$

no heat is gained or lost by system (all Δenergy = work done)

Use 1st Law: $dE + P\,dV = T\,dS$; For isolated, adiabatic expansion of volume, $dS = 0$.

$\Rightarrow \frac{4}{3}\pi c^2 (3a^2\dot{a}\rho + a^3\dot{\rho}) + P(4\pi a^2 \dot{a}) = 0$

$\Rightarrow 4\pi \left[3a^2\dot{a}\rho \frac{c^2}{3} + \frac{c^2}{3} a^3 \dot{\rho}\right] + 4\pi (P a^2 \dot{a}) = 0$

$\Rightarrow \dot{\rho} a^3 \frac{1}{3} c^2 + a^2 \dot{a} \rho c^2 + P a^2 \dot{a} = 0$

$\Rightarrow \dot{\rho} a^3 \frac{1}{3} c^2 + a^2 \dot{a} (\rho c^2 + P) = 0$

$\Rightarrow \dot{\rho} a^3 c^2 + 3a^2 \dot{a} (\rho c^2 + P) = 0$

$$\Rightarrow \dot{\rho}\frac{\cancel{a^3}\cancel{c^2}}{\cancel{a^3}\cancel{c^2}} + \frac{3\dot{a}\cancel{a^2}}{a^3 c^2}(\rho c^2 + P) = 0$$

$$\Rightarrow \boxed{\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0}$$

Acceleration Equation Derivation

start with Friedmann Equation: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$

$\Rightarrow \quad \dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - kc^2$

Differentiate with respect to time:

$$2\dot{a}\ddot{a} = \left(\frac{8\pi G}{3} a^2 \dot{\rho} + \frac{8\pi G}{3}\rho (2a)\dot{a}\right) - 0$$

[2nd derivative]

$$= \frac{8\pi G}{3}\left(\dot{\rho} a^2 + 2\rho a \dot{a}\right)$$

Fluid Equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

$$\Rightarrow \dot{\rho} = -\frac{3\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)$$

Plug fluid equation in:

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left[-3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right)a^2 + 2\rho a \dot{a}\right]$$

$$\Rightarrow 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left[-3\dot{a}a\rho + -3\dot{a}a\frac{P}{c^2} + 2\rho a\dot{a}\right]$$

$$\Rightarrow 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(-\rho a\dot{a} - 3\dot{a}a\frac{P}{c^2}\right)$$

$$= -\frac{8\pi G}{3}\left(\rho a\dot{a} + 3\dot{a}a\frac{P}{c^2}\right)$$

$$\Rightarrow \left[\frac{1}{2}\frac{1}{\dot{a}}\frac{1}{a}\right]2\dot{a}\ddot{a} = \left[\frac{1}{2}\frac{1}{\dot{a}}\frac{1}{a}\right]\left(-\frac{8\pi G}{3}\right)\left(\rho a\dot{a} + 3\dot{a}a\frac{P}{c^2}\right)$$

$$\Rightarrow \frac{\ddot{a}}{a} = \left(-\frac{4\pi G}{3}\right)\frac{1}{\dot{a}a}\left(\rho a\dot{a} + 3a\dot{a}\frac{P}{c^2}\right)$$

$$\Rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right)}$$

Redshift Derivation

$z = \frac{v}{c}$

Begin with the Robertson-Walker Metric:

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

light propogation obeys $ds=0$ <light travels no distance in space-time>

<at any point in time, all points in space are equivalent>

$\Rightarrow d\theta = d\phi = 0$

Now we have the RW metric as: $0 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + 0 + \ldots\right.$

$$\Rightarrow c^2dt^2 = \frac{a^2(t)\,dr^2}{(1-kr^2)}$$

sqrt B.S:

$$\Rightarrow c\,dt = \frac{a(t)\,dr}{\sqrt{1-kr^2}}$$

With the photons arrival time as t_0 and its emission time as t_e we can write:

$$\Rightarrow \int_{t_e}^{t_0} \frac{dt}{a(t)} = \frac{1}{c}\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}}$$

Consider a light ray emitted a short interval later therefore $t_e + dt_e$ and arriving a bit later too — $t_0 + dt_0$

$$\Rightarrow \int_{t_e+dt_e}^{t_0+dt_0} \frac{dt}{a(t)} = \frac{1}{c}\int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}}$$

Equating the two integrals:

$$\Rightarrow \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+dt_e}^{t_0+dt_0} \frac{dt}{a(t)} \underset{\text{rearrange limits}}{\Rightarrow} \int_{t_0}^{t_0+dt_0} \frac{dt}{a(t)} = \int_{t_e}^{t_e+dt_e} \frac{dt}{a(t)}$$

If we assume $a(t)$ is unchanging in these small intervals, we can take out integral:

$$\Rightarrow \frac{dt_0}{a(t_0)} = \frac{dt_e}{a(t_e)}$$

[Def of z]

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{\lambda_{obs}}{\lambda_{em}} - 1$$

Now the 2 seperate rays imagine are actually successive crests of a single light wave. Since $\lambda \propto$ time between crests : $\lambda \propto dt \propto a(t)$. Rearranging eqn

$$\Rightarrow z + 1 = \frac{\lambda_0}{\lambda_e} = \frac{dt_0}{dt_e} = \frac{a(t_0)}{a(t_e)}$$

On the Geometry of the Universe

Introduction

The Metric describes the physical distance between points, and is used to understand distances.

Begin with our knowledge on the distance *ds* between two points on a flat surface, given by Pythagoras's Theorem: $ds^2 = dx_1^2 + dx_2^2$, where dx_1 and dx_2 are the separations in the x_1 and x_2 coordinates.
If space is expanding, we then have the scale factor such that

$$ds^2 = a(t)(dx_1^2 + dx_2^2),$$

In general relativity however we are concerned with not only the spatial distance between points but the distance between points in four-dimensional space–time (which may also be curved).

Thus, we can write the distance as:

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu \, dx^\nu$$

where $\boldsymbol{g_{\mu\nu}}$ **is the metric** (i.e. the geometry of space-time is specified by this)**,** μ and ν are indices from 0 to 4, in which x^0 is the time coordinate, and the rest are the three special coordinates.

The metric comes as a solution from *Einstein's Field Equations*. One such solution—particularly important to cosmology—is the **Friedmann–Lemaître–Robertson–Walker (FLRW) metric**. This results by imposing the cosmological principle and including the possible curvatures.

(FLRW) metric:

$$ds^2 = -c^2 dt^2 + \; a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\right]$$

***Derivation separate page**

Before I will explain the *Einstein's Field Equations,* I believe it would appropriate to cover at least some of the basics mathematics that apply to it, most importantly—tensors. I will begin with their predecessor: vectors.

On Four-Vectors

A four-vector is another mathematical object with four components, which follows specific rules when a Lorentz transformation is applied to it.
We can define the *space-time* four-vector as $x^\mu, \mu = 0, 1, 2, 3$ as:

$x^0 = ct;\ x^1 = x;\ x^2 = y;\ x^3 = z$; Or in other notation as: $\vec{R} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$

We can define the *energy-momentum* four-vector $p^\mu, \mu = 0, 1, 2, 3$ as:

$p^0 = E;\ p^1 = p_x c;\ p^2 = p_y c;\ p^3 = p_z c$; Or in other notation as: $\vec{P} = \begin{bmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{bmatrix}$

The *Lorentz Transformation* of the Four-vectors appears in matrix form as:

$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma ct - \beta\gamma x \\ -\beta\gamma ct + \gamma x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{bmatrix} = \begin{bmatrix} \gamma E - \beta\gamma p_x c \\ -\beta\gamma E + p_x c \\ p_y c \\ p_z c \end{bmatrix}$ where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

or in another way as:

$$x^{0\prime} = \gamma(x^0 - \beta x^1);\ x^{1\prime} = \gamma(x^1 - \beta x^0);\ \ x^{2\prime} = x^2\ ;\ x^{3\prime} = x^3$$

These can be written more compactly as:
$x^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda_\nu^\mu x^\nu$, $\mu = 0, 1, 2, 3$ and the coefficient Λ_ν^μ is simply the elements of the matrix Λ:

$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (So $\Lambda_1^1 = \gamma;\ \Lambda_0^1 = \Lambda_1^0 - \beta\gamma;\ \Lambda_2^2 = \Lambda_3^3 = 1$)

Einstein Summation Convention says that repeating indices (occurring twice as a subscript, twice as a superscript, or once as a subscript and once as a superscript) will be summed from 0 to 3., thereby avoiding writing a lot of sigma's. So we can write in a "new" way as:

$$x^{\mu\prime} = \Lambda_\nu^\mu x^\nu$$

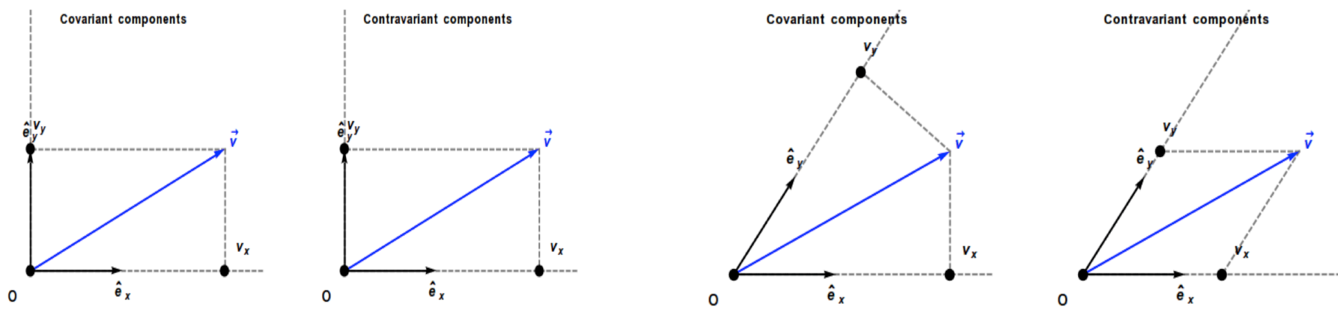
When changing from one coordinate system to another, there are a few of the $x^{\mu\prime}$ which do not change. This quantity is an *invariant* (as it does not vary under the transformation, for example as $r^2 = x^2 + y^2 + z^2$ is invariant under rotations):

$I \equiv (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = (x^{0\prime})^2 + (x^{1\prime})^2 + (x^{2\prime})^2 + (x^{3\prime})^2$
Because of the minus signs we can introduce the *metric* $g_{\mu\nu}$ as a matrix:

$$\boldsymbol{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus the $diag(1, -1, -1, -1) = diag\ (g_{00}, g_{11}, g_{22},\ g_{33})$.

The four-vector we were discussing is defined more specifically as the *contravariant four-vector.* We can introduce the *Covariant Four-Vector* by defining it as:
$x_{\mu} = g_{\mu\nu}x^{\nu}; (x_0 = x^0, x_1 = -x^1, x_2 = -x^2, x_3 = -x^3)$.
So the covariant vector is obtained by switching the signs of the contravariant vector.



The above images shows how the vectors start from the same positions but move differently as they transform. The *covariant* makes an arc-shape with the v_y component while v_x component remains unchanged while the *contravariant* has both components moving straight across with each component moving opposite to one another.

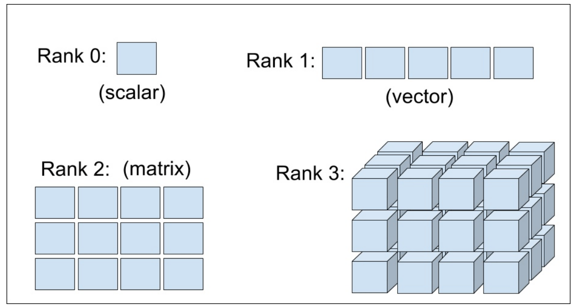
To clarify this means that the difference between the two is that the contravariant vector (usually simply called vector) transforms opposite to how the basis vectors transforms while the covariant vector (also called dual vector) will transform just like the basis vectors. I imagine it as if I were to point my arm in front of me and rotate left keeping my arm in front of me my arm will rotate in the same direction (i.e. covariant) but if I were to leave my arm pointing in the same direction but turn my body right independently from my arm, then my arm "rotates" in the opposite direction (contravariant).

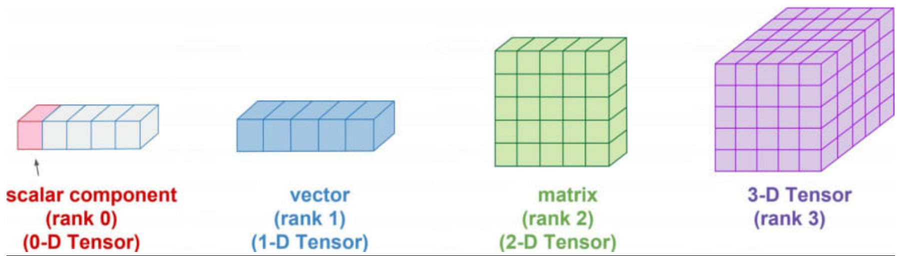
{*See Contravariant and Covariant Tensors*}

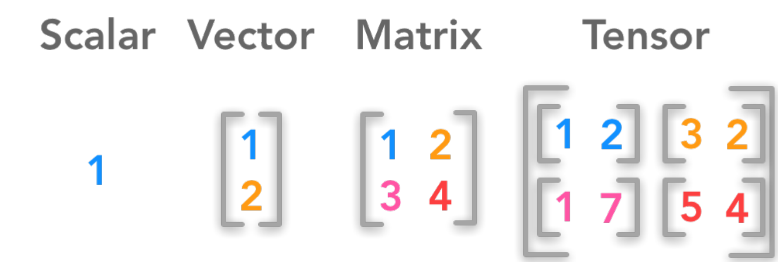
<u>On Tensors</u>

The tensor a mathematical objects which succeed our understanding of *scalars, vectors,* and *matrices*. They are generalizations of these quantities. The scalar is a quantity that remains invariant under rotations of a coordinate system and so can be represented by a single real number. The vector is a quantity that can be specified by multiple real numbers, which represent the dimension of the coordinate system, with the components transforming, like the coordinates of a fixed point when the coordinate system is rotated. The scalars can be called *tensors of rank 0*, while vectors can be called *tensors of rank 1*.

Visualizations of Tensors:







A tensor of rank of rank n in a d-dimensional space has two important properties:

1. It has n indices. Each index goes from 1 to d, thus having d^n total components.
2. These components transform in a specific way when applying a coordinate transformation.

Another way to explain tensors is as follows:
Let there be a vector **A** and a vector **B** in an *isotropic* medium. Let them be related by vector equation **A** = k**B** for a constant k, with **A** and **B** having the same direction. If the medium is isotropic then no matter which direction the vector points it will have the same value. Thus, all around, the vector equation will be satisfied. If however, the medium is not isotropic then the scalar k must be replaced with another mathematical object that will change the magnitude and direction of the vector that it is acting on. This object is the *tensor*.

Covariant & Contravariant Tensors

Since tensors are second rank vectors, we can represent it similar to the four vectors but now they have two indices and two factors of lambda:
$s^{\mu\nu\prime} = \Lambda^{\mu}_{\kappa}\Lambda^{\nu}_{\sigma}\, s^{\kappa\sigma}$; will have $4^2 = 16$ components as opposed to the vectors $4^1 = 4$ components. And a tensor of rank 3 will have $4^3 = 64$ components and with three factors of lambda.

On Einstein's Field Equations

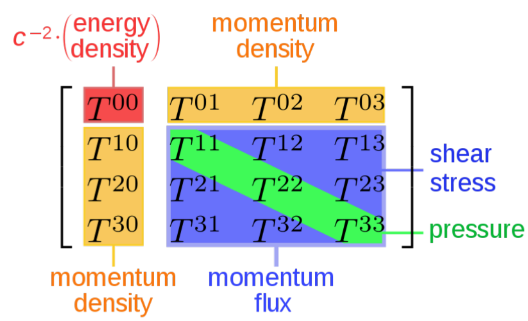
Einstein's Field Equations is a set of 16 partial differential equations that relate matter and the geometry of space-time. They describe the gravitational effects produced by a mass in the framework of relativity. Due to the symmetry of $T_{\mu\nu}$ reduces the number of equations to 10. It is given below:

$$R^{\mu}_{\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

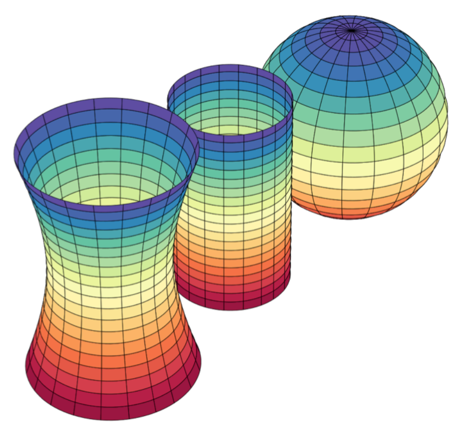
~~*Derivation separate page~~

Short description of each term:

The *Stress-Energy Tensor* ($T_{\mu\nu}$) (or Energy-Momentum tensor) tells you how the energy (which includes entities with mass, momentum, pressure—everything is energy) is distributed in the universe. In other words, it tells you the density and flux of the energy in the universe.



The *Einstein Tensor* ($G_{\mu\nu}$) is a combination of a few terms that together describes the physical geometry of spacetime. They work in the framework of *Riemannian Geometry* (also called Elliptical Geometry). This geometry rejects Euclid's V postulate, which can be stated as *Playfair's Axiom: In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point*. Riemannian Geometry has no parallel lines and can extend any straight line continuously without bounds. It studies smooth spaces (called manifolds).



Surface of negative Gaussian Curvature (hyperboloid), zero Gaussian curvature (cylinder) and positive Gaussian curvature (sphere). The Riemann Curvature Tensor describes the different curvatures in different directions.

Riemann Curvature Tensor (R^{μ}_{ν}) is the most important term describing the curvature of space. It tells you how matter will tend to converge or diverge. If it is equal to zero, then space-time is flat. In 4 dimensions, it takes 20 numbers to specify the curvature at each point. 10 of these numbers are captured by the Ricci tensor, while the other 10 are by another tensor called Weyl tensor.

The *Ricci Curvature Scalar (R)* is the simplest description of curvature but is closely related to the Curvature tensor. It represents the amount by which the volume of a geodesic ball in Riemannian manifold changes compared to a ball in flat space.

The *Cosmological Constant* (Λ) is a term added to describe some kind of energy ("vacuum energy") in the universe, which drives its acceleration. This energy has been called "dark energy" as it does not interact (only extremely rarely) with regular energy forms and therefore difficult to observe.

The *metric tensor* ($g_{\mu\nu}$) looks at the geometry of manifolds (i.e. space). The example that was described earlier revealed that in the flat 2D case we have:
$ds^2 = dx_1^2 + dx_2^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu \, dx^\nu$ with this case $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(Christofell Symbols are calculated from the metric tensor!) ***(*Separate Page)***

Geodesic

Another important concept that intimately ties in with general relativity and Einstein's Field Equations is the idea of the *Geodesic.* In simple terms, it is the path that a particle travels by in spacetime. The straight line —the shortest possible distance from one point to another in flat space—is equivalent to the geodesic in curved spacetime.

SUMMARY

In summary, the Einstein Field Equations illustrate the behavior of spacetime—with the *Einstein tensor* consisting of *Riemann Tensor* and *Ricci scalar* describes the curvature and geometry of spacetime while the *Stress-Energy Tensor* discusses the material and energy present in the universe. Each entity affects one another. As famous physicist John Wheeler put it "Space-time tells matter how to move; matter tells space-time how to curve".

The solutions to the Einstein Field Equations are the components of the metric tensor. The trajectories of the particles in this geometry are calculated using the geodesic equation—which tells us how free-falling matter and energy moves through spacetime. Together, these equations form the core and essence of The Theory of General Relativity.

Thesis Main Project:

Varying-G Cosmology

Exploring Varying-*G* and Λ Cosmology

Yury Chernyak

Abstract

This paper explores an alternative idea that the physical constant *G* (gravitational constant) is time-variant. By analyzing and studying compiled type Ia supernovae (SNe Ia) data and their redshifts, I shall attempt to use this information in developing models in which gravity varies in time as the universe evolves. The models, in which the strength of gravity must increase over time, should fit well with the type Ia supernovae, however the bounds will not be satisfied causing a failure of these models. The results will be analyzed and discussed, and then an analysis will be carried out regarding the possibility of varying *G* in context with General Relativity. I will discuss theoretical implications of this possibility that includes the requirement of a varying Λ term (the constant in Einstein's Field Equation).

I. Introduction

In 1998, two independent research teams—the High-Z team and the Supernova Cosmology Project--independently determined that type Ia supernovae appear fainter than what was expected leading them to conclude that the type Ia supernovae are further than expected therefore the average rate of expansion of the universe must be greater than what has been previously assumed. This results in today's accepted hypothesis is that the universe is accelerating, which is thought to be due to a form of energy known as dark energy—identified with the cosmological constant from Einstein's Field Equations.

Einstein's Field Equation:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Friedmann Equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

The standard model of cosmology has that k = 0 (representing a flat universe; k represents spatial curvature) and Λ > 0. However, the current theory predicts Λ to be at least 10^{50} times the observed value. I shall consider varying-G models by imposing the parameter that k = Λ = 0.

I begin creating a model with the Friedmann Equation (above) along with its associated Friedmann-Lemaitre-Robertson-Walker (FLRW) metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

. *{Derivation in Appendix}

The Friedmann Equation describes the expansion of the Universe while the FLRW metric is an exact solution to Einstein's field equation describing the geometry of a homogenous and isotropic universe (two main assumptions of cosmology). In these equations, *a(t)* is the dimensionless scale factor, normalized so that $a_0(t) = 1$ at present time $t = t_0$, $\dot{a} = \frac{da}{dt}$, G is the gravitational constant, ρ is the density of all forms of energy, *r* is the radial coordinate, *c* is the speed of light (constant), and *K* is the spatial curvature.

Setting up the model, I will introduce the *comoving distance* and *proper distance*. Proper distance is the distance measured in cosmology where a distant object would be at a specific moment in time. This distance can change in time as a result of the expansion of the universe.

The comoving distance factors out the expansion of the universe and thus results in a distance that does not change in time. In this project, I will be working with the comoving distance. It is defined as: $\chi = \int_0^r \frac{1}{\sqrt{1-Kr'^2}} = \sin^{-1}\frac{\sqrt{K}r}{\sqrt{K}}$ with the proper distance at time *t* being $d(t) = a(t)\chi$ (today they are equal since $a(t_0) = 1$). The comoving distance equation inverted results in

$r = \frac{\sin(\sqrt{K}\chi)}{\sqrt{K}}$ *{*Problem A2.2 in Appendix*}

II. Supernovae Cosmology

The first problem in cosmology that must be tackled is being able to measure distances to galaxies. I begin with a "standard candle", that is, a source whose luminosity is known. The most commonly used standard candles are the Type Ia Supernovae. A supernova is a very gigantic explosion of a star in space. A type Ia supernova occurs in a binary system. There is some star that exists in a gravitational system with a white dwarf (a small and very dense star, which are regarded as the final state of the life of a star). What occurs is the matter of the star is released and begins to accrete (i.e. accumulated) onto the white dwarf, until the white dwarf becomes so dense that it goes over a limit (the Chandrasekhar limit) thereby causing a nuclear burning (fusion) resulting in an explosion. To get an idea on how powerful a supernova of this type is—such a supernovae will release more energy in weeks than the Sun does in 10 billion years.

A. Luminosity Distance

The luminosity distance is the distance that an object appears to have (assuming a $\frac{1}{r^2}$ for the reduction of light intensity with distance holds true). It is a way to express the amount of light received from a distant object. Since the proper distance between two points is space is too large for us to measure, cosmologists and astronomers gather the data of the light received from a luminous object like a supernova and relate it to the *energy flux*. They then use this to extrapolate information on the approximate distance of the object—known as *luminosity distance.*

Energy flux = $f = \frac{L}{4\pi r^2}$, where L is the objects luminosity (i.e. the rate of total energy emission $L = \frac{dE}{dt}$). However, due to the expansion of the universe, the luminosity of an object decreases by the factor $(1+z)^2$, in which z is defined as the redshift. Redshift is the phenomenon where energy emitted from an object has an increase in wavelength as it travels through great distances in space. This is similar to the Doppler effect—the changing of frequency of a wave in relation to an observer who is moving relative to the source—applied to cosmology. Therefore, the corrected expression for flux is:

$f = \frac{L}{4\pi[(1+z)r]^2} = \frac{L}{4\pi d_L^2}$, in which the *luminosity distance* is defined as $d_L \equiv (1+z)r$.

B. Angular Diameter Distance

The angular diameter distance is defined as the distance that an object of known physical size appears to be at assuming a Euclidean geometry of space. It is a measure of how large objects appear to be. Similar to luminosity distance is that it too depends on redshift, however has a different dependence. Its equation is given as follows:

$d_{diam} = \frac{a_0 r_0}{(1+z)}$.

So this equation tells us that the size of objects looks bigger due to redshift and therefore objects appear closer than what they really.

*{*Code & Graphs in Appendix*}

III. Varying-G Friedmann Equation

Using the parameter that $K = \Lambda = 0$ reduces the (dimensionless) Friedmann equation (below). I shall assume that the Friedmann equation with this parameter of the universe will allow a G that varies with time. This works however only as an approximation for the Friedmann equation. This is because the Friedmann equation is derived from Einstein's field equations, which actually does not allow variations in *G*. To allow the possibility of a varying *G*, rivaling theories such as scalar-tensor theories have been proposed. However I will discuss at the end of the paper how it may be possible to vary-*G* in accordance with General Relativity.

*{***See Pre-Thesis Introduction:*** **On The Geometry of the Universe**}

My assumption will result in a modified Friedmann equation with a time-dependent *G* along with additional terms of order $\frac{\dot{G}}{G}$. If these terms are small, I can ignore them such that the Friedmann equation would become identical to the standard one but a time-varying *G*.

Let $G(t) = G_0 f(a)$, where G_0 is today's value and $f(a)$ is a function describing the dependence of *G* on the scale factor and therefore explaining how it has evolved in time.

Definitions & Constants

I discuss the meaning of these variables in *Brief Summary of Cosmology* (Pre-Thesis Section). **Def. 1** is the critical density of today—the mass of material in the universe per volume when the universe is flat ($k = 0$). Cosmologists relate the density of different materials ($\rho_\Lambda, \rho_m, \rho_r$—dark energy, matter, and radiation, respectively), to this critical density value, which is represented by the omega terms (Ω_Λ, Ω_m). Def. 6 represent today's omega value, which takes into account matter and dark energy (radiation is considered an important quantity earlier in the universe's history but not so much today—we say that today is a "matter dominated" universe). But since we consider the parameter that $\Lambda = 0$, we result with $\Omega_\Lambda = 0$ and therefore **Def. 6** results in $\Omega_0 = \Omega_{m,0} = 1$.

Def. 1: $\rho_{c,0} \equiv \frac{3H_0^2}{8\pi G_0}$ **Def. 2:** $\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G_0}$ **Def. 3:** $\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_{c,0}}$

Def. 4: $\Omega_m = \frac{\rho}{\rho_{c,0}}$ **Def. 5:** $\Omega_m = \frac{\Omega_{m,0}}{a^3}$; $\Omega_{m,0} \approx 1$ **Def. 6:** $\Omega_0 \equiv \Omega_m + \Omega_\Lambda$

Def. 7: $z \equiv \frac{a_0}{a} - 1$ (Redshift)

Const. 1: $\Omega_\Lambda = 0$

Const. 2: $\Omega_0 = 1$

Const. 3: $G_0 = 6.672 \times 10^{-11}$ m^{3}kg^{-1}sec^{-1}

Const. 4: $c = 2.998 \times 10^8$ m sec^{-1}

The dimensionless Friedmann Equations can be written as:

$H^2 = \left(\frac{\dot{a}}{a}\right) = H_0^2[\,\Omega_m(a) + (1-\Omega_0)a^{-2} + \Omega_\Lambda]$ With $-Kc^2 = H_0^2(1-\Omega_0)$

This results in:

$$r(z) = cH_0^{-1}\int_0^{1+z}\frac{da}{a^2\sqrt{\,\Omega_m(a) + (1-\Omega_0)a^{-2} + \Omega_\Lambda}}$$

$$t(a) = H_0^{-1}\int_0^{a}\frac{da}{a\sqrt{\,\Omega_m(a) + (1-\Omega_0)a^{-2} + \Omega_\Lambda}}$$

*{*See Derivation and Problem A2.4 in Appendix*}

The result is similar when I include the varying *G*-component in the dimensionless Friedmann Equation. The results become:

$$r(z) = cH_0^{-1}\int_0^{1+z}\frac{da}{a^2\sqrt{f(a)\,\Omega_m(a) + (1-\Omega_0)a^{-2} + \Omega_\Lambda}}$$

$$t(a) = H_0^{-1}\int_0^{a}\frac{da}{a\sqrt{f(a)\,\Omega_m(a) + (1-\Omega_0)a^{-2} + \Omega_\Lambda}}$$

These are the modified Friedman equation with a time-varying *G*, whose influence is found in the function *f(a)*.

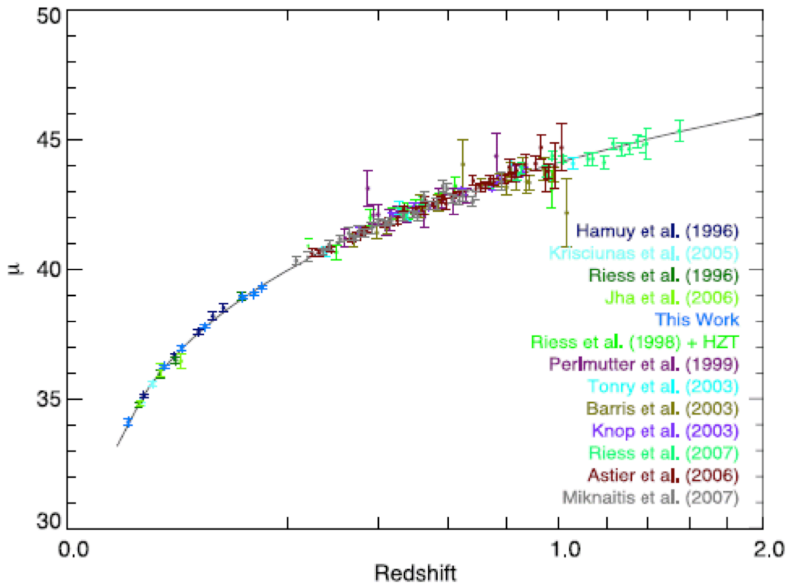
IV. Varying-G Models

I take two of the models from *Dungan* and *Prosper* paper that fit the supernova data.

Model 1: $f(a) = e^{b(a-1)}$ and **Model 2:** $f(a) = \frac{2}{(1+e^{-b(a-1)})}$

Where *b* is a dimensionless and adjustable parameter.

The supernova data, which was taken from *Kowalski*, is presented on the right:



The y-axis is called *distance modulus*. This is closely related to the concepts of luminosity distance discussed earlier. It is defined as:

$$d_{mod} \equiv 5 \log \left(\frac{d_L}{10^{-5}}\right)$$

I plug in and evaluate *model 1* into the modified Friedmann Equations to get:

*Reminder: $\Omega_m = \frac{\Omega_{m,0}}{a^3}$; $\Omega_{m,0} \approx 0.3$; $\Omega_\Lambda = 0$; $\Omega_0 = 1$

The Comoving Distance:

$$r(z) = cH_0^{-1} \int_{(1+z)^{-1}}^{1} \frac{da}{a^2\sqrt{e^{b(a-1)}\left(\frac{0.3}{a^3}\right)+0+0}} = cH_0^{-1}\left(e^{b/2}\sqrt{\frac{20\pi}{3b}}\,\mathrm{erf}\left(\sqrt{\frac{bx}{2}}\right)\right)\Bigg]_{(1+z)^{-1}}^{1}$$

$$= \frac{c}{H_0} e^{b/2}\sqrt{\frac{20\pi}{3b}}\left[\mathrm{erf}\left(\sqrt{\frac{b}{2}}\right) - \mathrm{erf}\left(\sqrt{\frac{b(1+z)^{-1}}{2}}\right)\right]$$

The Universal Time:

$$t(a) = cH_0^{-1}\int_0^a \frac{da}{a\sqrt{e^{b(a-1)}\left(\frac{1}{a^3}\right)+0+0}} = cH_0^{-1}\left(\sqrt{\frac{20\pi}{3b}}\,e^{b/2}\,\mathrm{erf}\sqrt{\frac{ba}{2}} - 2\sqrt{\frac{10a}{3}}\,e^{\left(\frac{-ba}{2}+\frac{b}{2}\right)}\right)/b$$

Model 2 integral can only be evaluated numerically using numerical methods of approximation.

The model 1 fit will give a value $b = 2.09 \pm 0.08$ when fitted onto the supernova data. This model is evaluated to give a universal time of $t_0 \approx 15.1 \times 10^9\ years$. For model 2, $b = 3.27 \pm 0.11$ and $t_0 \approx 16.2 \times 10^9\ years$.

The bounds are discussed in the next section.

V. Experimental Values and Limits of varying-G Universe

When comparing the calculated predictions with observational data I find that despite the fits to the supernova with these models, it does not satisfy the bounds. These models predict a different age of the universe (model 1 being 15.1 billion) compared to the currently accepted value of approximately 13.8 billion years old.

In the paper by *Dungan* and *Prosper* which I am following, they characterize $\frac{\dot{G}}{G}$ in terms of the logarithmic derivative of the function $f(a)$: $\frac{\dot{G}}{G} = \frac{1}{G}\frac{dG}{da}\dot{a} = H_o \frac{d(\ln f)}{da}$ in which they find that $\frac{\dot{G}}{G} = 1.5 \times 10^{-10} yr^{-1}$ when $f(a) = e^{b(a-1)}$ and $\frac{\dot{G}}{G} = 1.15 \times 10^{-10} yr^{-1}$ when $f(a) = \frac{2}{(1+e^{-b(a-1)})}$. When comparing these results to the ones in the table below, we can see that the bounds of model 1 and model 2 are one to three orders of magnitude larger than the upper bounds found in other experimental-values.

These results (from other authors) have been deduced from supernovae data as well as other methods. Most notable experimental results are:

- Gaztanaga et al. (2001) finds that the $\frac{\dot{G}}{G} \leq 10^{-11} yr^{-1}$ at redshifts of z ~ 0.5 when looking at the correlation between nickel synthesis in the outbursts and luminosity.
- Verbiest et al. (2008) measured orbital period rates of pulsars and set a limit of $\frac{\dot{G}}{G} = 23 \times 10^{-12} yr^{-1}$
- Corsico et al. (2013) conclude a white pulsation limit of $\frac{\dot{G}}{G} = -1.3 \times 10^{-11} yr^{-1}$
- Overall the current supernovae data (Suzuki et al. 2012) with ΛCDM cosmology concludes a range of limit of $\frac{\dot{G}}{G} = (-3, +7.3) \times 10^{-11} yr^{-1}$

Other limits based on other aspects of study are summarized in the table below:

Table 1. Values of α and β for average $\dot{G}/G$ when $t_0 = 14$ Gyr, $\Omega_m = 0.3, \Omega_\Lambda = 0.7$ and $z \simeq 0$

Ranges of $\dot{G}/G$ yr^{-1}	Sources	α	β
$-(1.10 \pm 1.07) \times 10^{-11} < \frac{\dot{G}}{G} < 0$	PSR $1913 + 16(Damouretal.1988)$	-0.0852	0.4074
$-1.60 \times 10^{-12} < \frac{\dot{G}}{G} < 0$	Helioseismological data (Guenther et al, 1998)	-0.0115	0.0670
$(-1.30 \pm 2.70) \times 10^{-11}$	PSR B1855+09 (Arzoumanian 1995; Kaspi, Taylor & Ryba 1994)	-0.1023	0.4698
$(-8 \pm 5) \times 10^{-11}$	Lunar occultation (Van Flandern 1975)	-1.333	1.6
$(-6.4 \pm 2.2) \times 10^{-11}$	Lunar tidal acceleration (Van Flandern 1975)	-0.8421	1.4328
-15.30×10^{-11}	Early Dirac theory (Blake 1978)	11.7692	2.0582
-5.1×10^{-11}	Additive creation theory (Blake 1978)	-0.5730	1.2644
$(-16 \pm 11) \times 10^{-11}$	Multiplication creation theory (Faulkner 1976)	8.00	2.0869
$-2.5 \times 10^{-10} \leqslant \frac{\dot{G}}{G} \leqslant +4.0 \times 10^{-11}$	WDG 117-B15A (Benvenuto et al. 2004)	-3.0	1.8
$\left\lvert \frac{\dot{G}}{G} \right\rvert \leqslant +4.10 \times 10^{-10}$	WDG 117-B15A [18]	1.1319	3.09
$-(0.6 \pm 4.2) \times 10^{-12}$	Double-neutron star binaries (Thorsett 1996)	-0.0043	0.0254
$(0.46 \pm 1.0) \times 10^{-12}$	Lunar Laser Ranging (Turyshev et al. 2003)	0.0318	-0.2110
$1 \times 10^{-11 \pm 1}$	Wu and Wang (1986)	0.0666	-0.5

(Ray & Mukhopadhyay, 2018)

VI. Discussion, Analysis and Varying-Λ possibility

I mentioned earlier in this paper that varying-*G* was not possible in Einstein's Field Equation's. Therefore, this paper makes the wrong assumption by not taking this important fact into account. However exploring further, we have the idea that solving the *Bianchi Identities*—a specific *covariant derivative of the Riemann Tensor* in which it equals zero—would result in a fluid equation that includes a varying-*G* component. Moreover, the varying-*G* fluid equation must include a varying-Λ term in order to make sense. In other words a varying-*G* would imply a varying-lambda. If this were the case, General Relativity would incorporate both terms as time-variant and avoid the issue of breaking the law of conservation of energy.

Bianchi Identities:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0$$

in which the covariant derivative of the Riemann Tensor is:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\partial_\beta\partial_\mu g_{\nu\alpha} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}\right]$$

In other words, this is when the Einstein Tensor is set equal to zero — $\nabla_\mu G^{\mu\nu} = 0$

These can be solved using the same Christoffell Symbols that I had been working with when deriving the FLRW metric. *{*See Derivations in Appendix*}

Fluid Equation:

The Fluid Equation looks as follows: $\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$

However incorporating the varying-*G* and varying- Λ results in the fluid equation below:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) + \frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}c^2}{8\pi G} = 0, \quad \Rightarrow \quad \dot{\Lambda} = -\frac{8\pi\dot{G}}{c^2}\rho.$$

VII. Summary

Paul Dirac first suggested the idea of a variable G in 1937. Cosmological theories incorporating this idea had been developed a few decades later such as Brans-Dicke theory (1961) in which $[\phi(t)]^{-1} \propto G(t)$ where the former part is the scaler field and is increasing in time (this is a scalar tensor theory—an inferior challenger to the Theory of General Relativity), Hoyle-Narlikar theory (1972), and the theory of Dirac (1973). These theories viewed a gravitational constant to have decreased with time.

In order for this idea to function, it must be in accordance with the Theory of General Relativity as well as the Theory of an Expanding Universe. This has been shown for the former with the three theories mentioned above, as well as the theory of an expanding universe in which $\dot{G}/G = \sigma H_0$. Furthermore, superstring theory allows G to be a varying quantity (Marciano 1984).

Experimentally, a varying-G is supported by results from Lunar Laser Ranging

(Turyshev et al. 2003), spinning rate of pulsars (Arzoumanian 1995; Kaspi, Taylor & Ryba 1994; Stairs 2003), Viking Lander (Hellings 1987; Reasem- berg 1983), distant Type Ia supernova observation (Gaztanaga et al. 2002), Helioseismological data (Guenther et al. 1998), white dwarf G117-B15A (Biesiada & Malec 2004; Benvenuto et al. 2004).

However, this paper explored models in which the strength of gravity increases over time found that these models fit the type Ia supernova however were not consistent with the experimental data. This was perhaps a result because the *Dungan & Prosper* paper did not consider varying-lambda possibilities. Perhaps with different models of $f(a)$ and incorporating lambda, a different result may be obtained that would allow the formulation of a varying-*G* and a varying- Λ theory in cosmology.

VII. Comments on this Project

I had many challenges throughout the course of this project however I learned a lot as well. I was able to improve my skills in coding by incorporating the python program in this project. I had a much better understanding of General Relativity and Cosmology regarding both the physical understanding and the mathematics behind the physical interpretations. And I also acquired skills in breaking down a big problem into small components and organizing the various tasks at hand. The next steps in this project would be to solve the Bianchi Identities to see if the fluid equation would indeed result in the varying lambda and varying-*G* terms. Furthermore I would develop new models of *f(a)*, that is, fits to the supernova data incorporating varying-*G*. And I would incorporate the lambda term (that varies) in those modified Friedmann equations as well (and solve the resulting *t(a)* and *r(z)*). Lastly, I would consider the possibilities of different curvatures and different lambda combinations.

THE END

Appendix

Derivations, Code, & Problems

On Luminosity and Distances

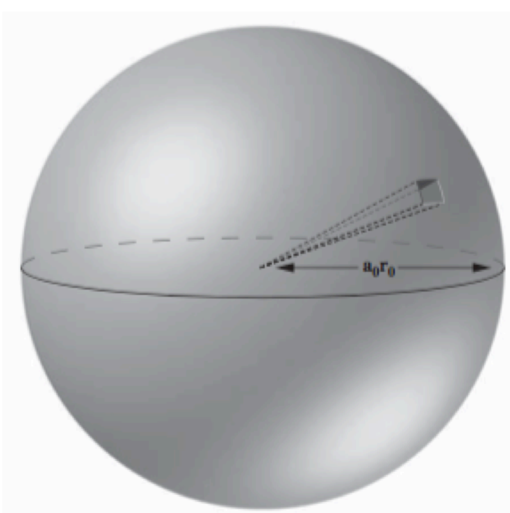
The cosmology metric—Robertson-Walker Metric shows the path flow in space-time. When discussing light propagation, the metric will represent this with $ds^2 = 0$, as light will travel no distance in space-time. From this fact, we derived the ***redshift*** formula (**separate page***).
Now I will present two other important quantities for deriving distances in cosmology: **luminosity distance** and **angular diameter distance**.

Luminosity Distance

This is defined as the distance that an object *appears* to be, assuming an inverse-square law $(1/r^2)$. However the universe does not need to follow this law as the geometry of the universe does not need to be flat (fortunately our observations indicate that it is, so no need to worry about this) and that our universe is expanding. As a consequence of the latter, we find that each individual photon loses energy $\propto (1+z)$ and photons have less frequency also $\propto (1+z)$.
Now we define *luminosity L* of an object as the energy emitted per unit solid angle per second. The radiation *flux density S* received by us from the object is defined as the energy per unit area per second.

So the *luminosity distance* is: $d_{lum}^2 = \frac{L}{S}$ with L/S being the unit area per unit solid angle.

The image below demonstrates this:



The light reaches us a distance of $a_0 r_0$ but the surface area at that point is $4\pi a_0^2 r_0^2$. Thus the radiation flux will be $S = {}^{L}/_{a_0^2 r_0^2}$ but taking into account the consequences of the expanding universe, we must then add two factors of the scale factor *a = (1+z)*.
Therefore, the more accurate radiation flux is:

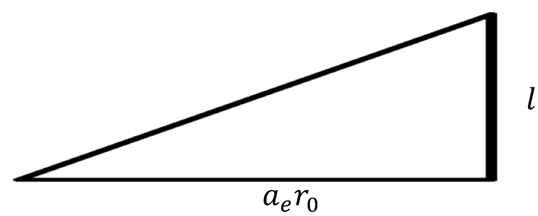
$$S = \frac{L}{a_0^2 r_0^2 (1+z)^2}$$

Thus, $d_{lum}^2 = a_0 r_0 (1+z)$
What this physically means is that objects appear further than what they really are because of that redshift effect, causing them to appear dimmer.

Angular Diameter Distance

The *angular diameter distance* is the distance that an object appears to be at and tells you how large the object appears to be.
It is given by the formula: $d_{diam} = \frac{l}{\sin\theta} = \frac{l}{\theta}$ where *l* is the known diameter/size of object, theta is the angle measured and d_{diam} is the distance between you and the object (i.e. how far away it is).

Now we have the physical size *l* is:



$l = \ (a_e r_0)\ d\theta$
(a_e) Is the scale factor at the time emitted

Which becomes:

$$d\theta = \frac{l}{a_e r_0} = \frac{l(1+z)}{a_0 r_0}$$

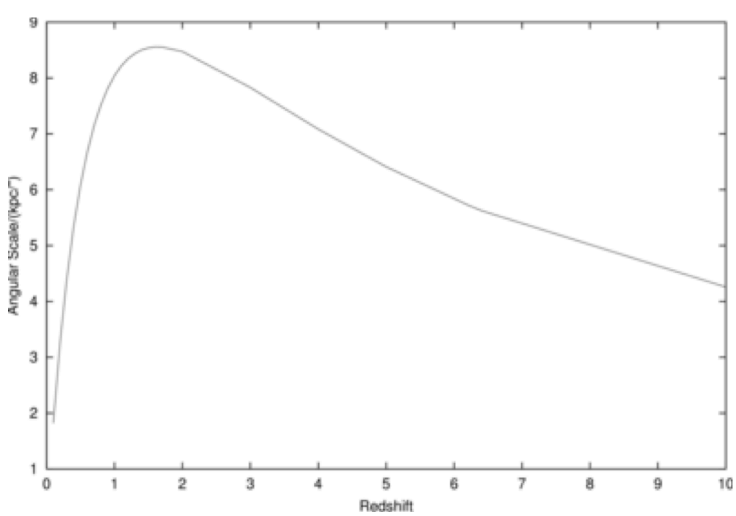
Where the $(1+z)$ factor accounts for the scale factor between emission time of light and present time.
Plugging the result of $d\theta$ into d_{diam} for θ we arrive at the formula:

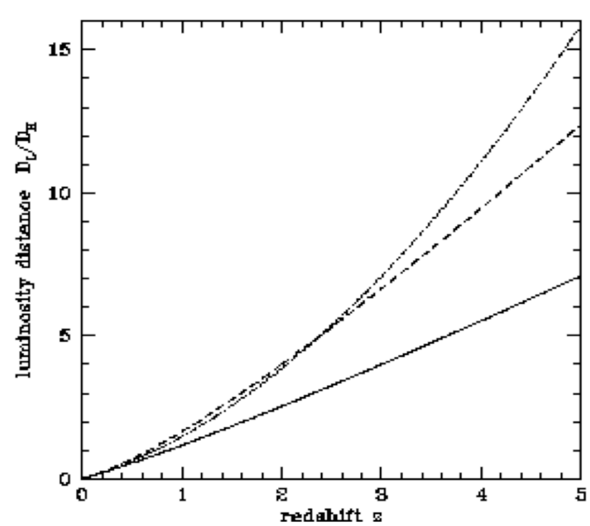
Angular diameter distance: $d_{diam} = \frac{a_0 r_0}{(1+z)}$

The physical meaning of this is that objects appear closer than what they really are and the size of objects look bigger because of redshifts.

In summary, close objects in space act as they would expect but going further out is when objects appear strange as the distant objects appear to be closer than what they really are. Those further objects are also dimmer (as the luminosity formula shows) and appear bigger (as the diameter formula shows that the angle gets larger making it seem that they spread over more space).

Their diagrams look as follows (which I had to replicate using Python):





Python Code

I wrote the code below which results in the following graphs representing *Luminosity Distance* and *Angular Diameter Distance*, both as functions of redshifts.

```
from scipy.integrate import quad

#FOR OMEGA = 1========================================

def r(z):
    cHo = 3000
    omega = 1
    integral = (cHo)*(1-omega+omega*(1+z)**3)**(-0.5)
    return integral

z_values = []    #contains all z values
r0_values = []   #this is the list that will contain all
                     #r_o values corresponding to z from 0.5 to 10

z = 0
while z <= 5:   #z values go from 0.5 to 10 by 0.5 increments
    i, error = quad(r, 0, z)  #evaluateS r(z) from 0-z,i = integral result
    r0_values.append(i)      #each i is a r_o value
    z += 0.1
    z_values.append(z)

d = []
for i in range(len(r0_values)):
    d_diam = r0_values[i]/(1+(z_values[i]))
    d.append(d_diam)


#FOR OMEGA = 0.3-----------------------------------
def r2(z):
    cHo = 3000
    omega = 0.3
    integral = (cHo)*(1-omega+omega*(1+z)**3)**(-0.5)
    return integral

r0_values_2 = []
z_values_2 = []

z = 0
while z <= 5:
    i, error = quad(r2, 0, z)
    r0_values_2.append(i)
    z += 0.1
    z_values_2.append(z)

d_2 = []
for i in range(len(r0_values_2)):
    d_diam = r0_values_2[i]/(1+(z_values_2[i]))
    d_2.append(d_diam)
```

Code continues on the right ➔

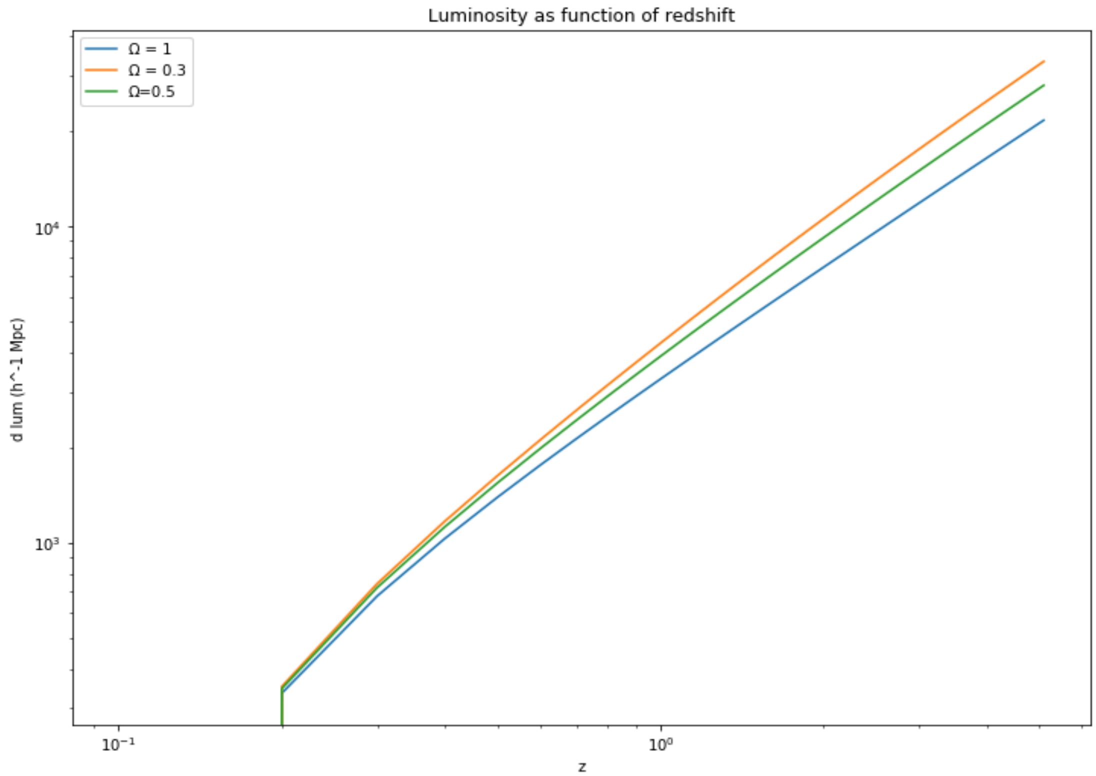
```
#FOR OMEGA = 0.5-----------------------------------
def r3(z):
    cHo = 3000
    omega = 0.5
    integral = (cHo)*(1-omega+omega*(1+z)**3)**(-0.5)
    return integral

r0_values_3 = []
z_values_3 = []

z = 0
while z <= 5:
    i, error = quad(r3, 0, z)
    r0_values_3.append(i)
    #print(i)
    z += 0.1
    z_values_3.append(z)

d_3 = []
for i in range(len(r0_values_3)):
    d_diam = r0_values_3[i]/(1+(z_values_3[i]))
    d_3.append(d_diam)


#GRAPHING-----------------------
import matplotlib.pyplot as plt
import numpy as np


x = z_values
y = d
x2 = z_values_2
y2 = d_2
x3 = z_values_3
y3 = d_3

w = 12
h = 9
d = 70
plt.figure(figsize=(w, h))

plt.plot(x, y)
plt.plot(x2, y2)
plt.plot(x3, y3)
plt.xscale('log')
#plt.yscale('log')
plt.grid(True)

plt.title('Diameter as function of redshift')
plt.xlabel('z')
plt.ylabel('d diam (h^-1 Mpc)')
plt.legend(['Ω = 1','Ω = 0.3', 'Ω=0.5'], loc='upper left')
```

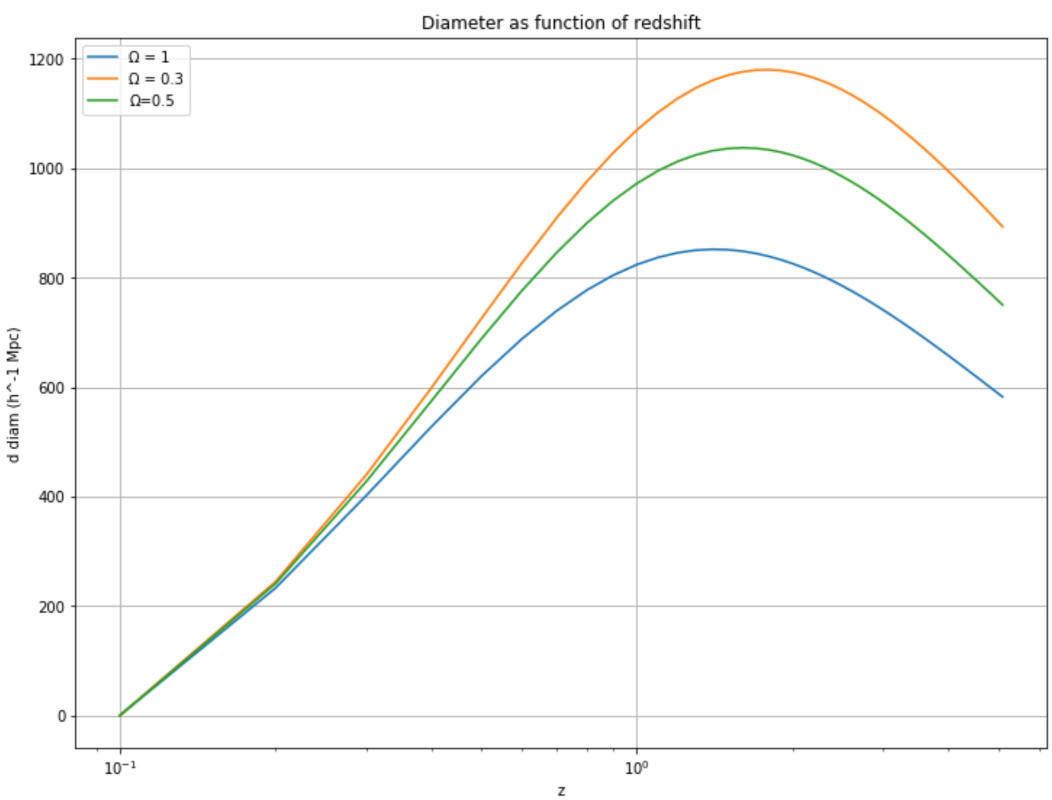
```
#=======PLOTTING LUMINOSITIES================

l = []  #luminosity values
for i in range(len(r0_values)):
    d_lum = r0_values[i]*(1+(z_values[i]))
    l.append(d_lum)

l_2 = []  #luminosity values
for i in range(len(r0_values_2)):
    d_lum = r0_values_2[i]*(1+(z_values_2[i]))
    l_2.append(d_lum)

l_3 = []  #luminosity values
for i in range(len(r0_values_3)):
    d_lum = r0_values_3[i]*(1+(z_values_3[i]))
    l_3.append(d_lum)
```

Continued Code on the right ➔

```
import matplotlib.pyplot as plt
w = 12
h = 9
d = 70
plt.figure(figsize=(w, h))

x_lum1 = z_values
y_lum1 = l
x_lum2 = z_values_2
y_lum2 = l_2
x_lum3 = z_values_3
y_lum3 = l_3

plt.xscale('log')
plt.yscale('log')
plt.plot(x_lum1, y_lum1)
plt.plot(x_lum2, y_lum2)
plt.plot(x_lum3, y_lum3)

plt.title('Luminosity as function of redshift')
plt.xlabel('z')
plt.ylabel('d lum (h^-1 Mpc)')
plt.legend(['Ω = 1','Ω = 0.3', 'Ω=0.5'], loc='upper left')
```

Python Graphs





Set of Problems

Problem 7.5

Show that in a spatially-flat matter-dominated cosmology the density parameter evolves as: $\Omega(z) = \Omega_0 \frac{(1+z)^3}{1-\Omega_0+(1+z)^3\Omega_0}$
If our universe has $\Omega_0 \cong 0.3$, *at what redshift did it begin accelerating?*

Problem A2.2

The present physical distance from the origin to an object at radial coordinate r_0 *is given by integrating* ds *at fixed time:*

$$d_{phys} = a_0 \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$$

Evaluate this.
Find an expression for d_{lum} *in terms of* d_{phys} *and* z.

Problem A2.4

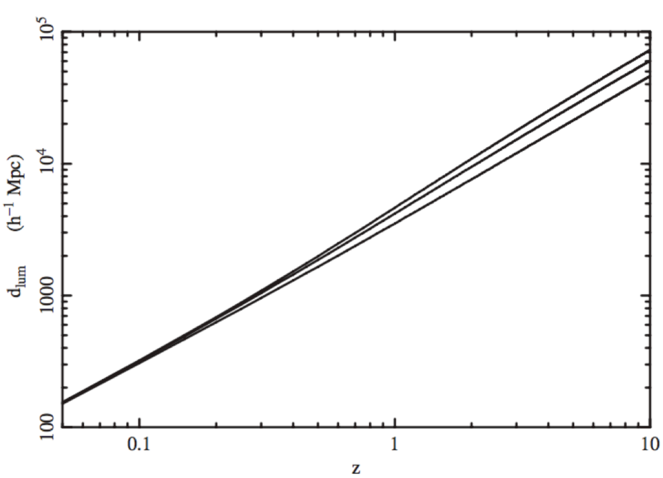
Demonstrate that for spatially-flat matter-dominated cosmologies with a cosmological constant the Friedmann equation can be written as:

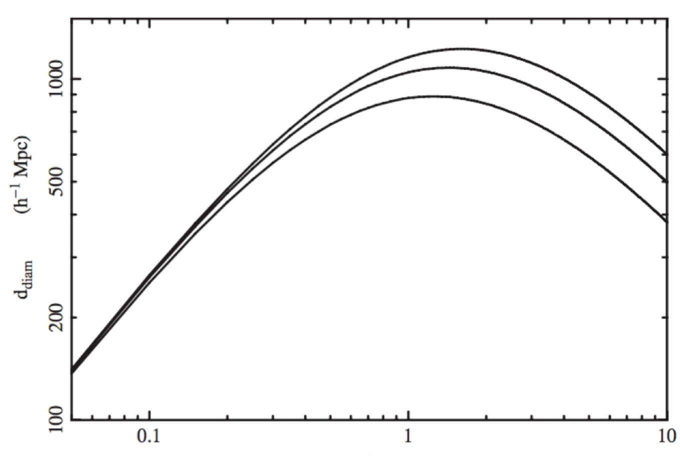
$$H^2(z) = H_0^2[1-\Omega_0+\Omega_0(1+z)^3]$$

Use this to show that for spatially-flat cosmologies:

$$r_0 = cH_0^{-1}\int_0^z \frac{dz}{[1-\Omega_0+\Omega_0(1+z)^3]^{1/2}}$$

Bearing in mind that $cH_0^{-1} = 3000h^{-1}Mpc$, *derive formulae for the luminosity and angular diameter distances as a function of redshift for the special case* $\Omega_0 = 1$. *Solve this equation numerically to obtain curves for* $\Omega_0 = 0.3$ *as shown below:*





Other Derivations/ Extra

1. *Friedmann Equation using Newtonian Method*
2. *Acceleration Equation*
3. *Fluid Equation*
4. *Redshift*
5. *FLRW Metric*
6. *Christofell Symbols*
7. ***Bianchi Identities***

\+ *On Geometry of Universe*, *On Luminosity & Distances*, *Summary of Cosmology* Documents

(Solutions are on Separate Scanned Pages)

Problem A.2.2

closed: $k > 0$ metric: $ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$

$$d_{phys} = a_0 \int_0^{r_0} \frac{dr}{\sqrt{1-Kr^2}}$$

$$= a_0 \left[\frac{\sin^{-1}(\sqrt{K}\,r)}{\sqrt{K}}\right]_0^{r_0} \qquad \left[\begin{array}{l} u = (1-kr^2)^{1/2} \\ du = \frac{1}{2}(1-kr^2)^{-1/2}(-2r) \end{array}\right]$$

$$= \frac{a_0}{\sqrt{K}} \sin^{-1}(\sqrt{K}\,r_0)$$

$$d_{lum} = a_0 r_0 (1+z)$$

$$= d_{phys}(1+z)$$

so, $$d_{lum\,(spherical)} = \frac{a_0 \sin^{-1}(\sqrt{k}\,r_0)}{\sqrt{k}}(1+z)$$

@ $z \ll 1$ $d_{lum} = d_{phys}$.

Problem A2.4

Begin with definition: $1+z = \frac{a_0}{a} \Rightarrow a \equiv \frac{1}{1+z}$

$\Rightarrow da = \left[(1+z)^{-1}\right]' = -(1+z)^{-2}\,dz$

$\Rightarrow da = \underbrace{\frac{-1}{(1+z)}}_{a}\underbrace{\frac{1}{(1+z)}}_{a}\,dz = -a^2\,dz = da$

$H \equiv \frac{\dot{a}}{a} = \frac{da/dt}{a} \Rightarrow aH = \frac{da}{dt} \Rightarrow \frac{da}{aH} = dt$

Now, divide Both sides by $a(t)$: $\frac{da/aH}{a(t)} = \frac{dt}{a(t)}$

$\Rightarrow \frac{da}{a(aH)} = \frac{da}{a^2 H} = \frac{-a^2\,dz}{a^2 H} = \frac{-dz}{H}$

so, $\boxed{\frac{dt}{a(t)} = \frac{-dz}{H}}$

$r_0 = \int_{t_e}^{t_0} \frac{c\,dt}{a(t)}$ plug into this! $\Rightarrow r_0 = \int_{t_e}^{t_0}\left(\frac{-dz}{H}\right)c$

our $H(z) = H_0\sqrt{\left[1-\Omega_0+\Omega_0(1+z)^3\right]}$

so, $r_0 = \int \frac{c\,(-dz)}{H_0\sqrt{1-\Omega_0+\Omega_0(1+z)^3}} \Rightarrow \boxed{r_0 = cH_0^{-1}\int_0^z \frac{dz}{\left[1-\Omega_0+\Omega_0(1+z)^3\right]^{1/2}}}$

Derivation of Universal Time

Friedmann Equation: $H(t)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a(t)^2}$ * set $c=1$

We know: $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$, $\Omega(t) = \frac{\rho(t)}{\rho_c(t)}$, $k = H_0^2(\Omega_0 - 1)$

F. Eqⁿ $\Rightarrow H(t)^2 = \frac{8\pi G}{3}\rho(t) - \frac{H_0^2(\Omega_0 - 1)}{a(t)^2}$

$$\Rightarrow \frac{H(t)^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\rho(t) - \frac{(1-\Omega_0)}{a(t)^2} = \frac{\rho(t)}{\rho_{c,0}} - \frac{1-\Omega_0}{a(t)^2} \quad ; \quad {}^{*}\frac{8\pi G}{3H_0^2} = \frac{1}{\rho_{c,0}}$$

We know: $\rho_m = \frac{\rho_{m,0}}{a^3}$; $\rho_r = \frac{\rho_{r,0}}{a^4}$, $\rho_{\Lambda,0}$ = constant.

$$\rho(t) = \rho_m + \rho_r + \rho_\Lambda \Rightarrow \frac{\rho(t)}{\rho_{c,0}} = \frac{\rho_m}{\rho_{c,0}} + \frac{\rho_r}{\rho_{c,0}} + \frac{\rho_\Lambda}{\rho_{c,0}} = \frac{\rho_{m,0}}{a^3\rho_{c,0}} + \frac{\rho_{r,0}}{a^4\rho_{c,0}} + \frac{\rho_{\Lambda,0}}{\rho_{c,0}}$$

we know also: $\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{c,0}}$; $\Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{c,0}}$; $\Omega_{\Lambda,0} = \frac{\rho_\Lambda}{\rho_{c,0}}$

$$\Rightarrow \frac{H^2}{H_0^2} = \left[\frac{\rho_{m,0}}{\rho_{c,0}}\frac{1}{a^3} + \frac{\rho_{r,0}}{\rho_{c,0}}\frac{1}{a^4} + \frac{\rho_{\Lambda,0}}{\rho_{c,0}}\right] + \frac{(1-\Omega_0)}{a^2} = \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_{\Lambda_0} + \frac{(1-\Omega_0)}{a^2}$$

We know: $H = \frac{\dot{a}}{a} \Rightarrow \frac{H^2}{H_0^2} = \frac{\dot{a}^2}{H_0^2 a^2}$

$$\Rightarrow \frac{\dot{a}^2}{H_0^2 a^2}\cdot a^2 = a^2\left(\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_{\Lambda_0} + \frac{(1-\Omega_0)}{a^2}\right) = \frac{\Omega_{m,0}}{a} + \frac{\Omega_{r,0}}{a^2} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)$$

Square root both sides and inverse.

$$\frac{H_0}{(da/dt)} = \frac{1}{\sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)}}$$

$\Longrightarrow$

Integrate:

$$\Rightarrow \int_0^t H_0\, dt = \int_0^a \frac{da}{\sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1-\Omega_0)}}$$

$$\Rightarrow t(a) = \frac{1}{H_0} \int_0^a \frac{da}{\sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0} a^2 + (1-\Omega_0)}} \quad (*)$$

We know: $da = \frac{da}{dz} dz$; $a = \frac{1}{1+z} \Rightarrow \frac{da}{dz} = -\frac{1}{(1+z)^2}$

plug these in integral above:

$$\Rightarrow t(z) = \frac{1}{H_0} \int_z^\infty \frac{dz}{(1+z)\left[\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0} + (1-\Omega_0)(1+z)^2\right]^{1/2}}$$

Derivation of FLRW METRIC

The line element of a ~~~~ flat spacetime describing distance between points is $ds^2 = -cdt^2 + dx^2 + dy^2 + dz^2$ also known as Minkowski space.

Spatial part can be transformed into spherical polar coordinates:

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

Differentiating: $(fgh)' = f'gh + fg'h + fgh'$

$dx' = d(r\sin\theta\cos\phi)'$

$= r'\sin\theta\cos\phi + r(\sin\theta)'\cos\phi + r\sin\theta(\cos\phi)'$

sin → cos
cos → −sin

$y' = (r\sin\theta\sin\phi)'$

$(fg)' = f'g + g'f$

$= r'\sin\theta\sin\phi + r(\sin\theta)'\sin\phi + r\sin\theta(\sin\phi)'$

$z' = (r\cos\theta)' = r'\cos\theta + r\cos\theta'$

$\Rightarrow dx = dr\sin\theta\cos\phi + r(\cos\theta\, d\theta\cos\phi) + r\sin\theta(\sin\phi\, d\phi)$

$dx^2 = (2r\,dr)\sin^2\theta\cos^2\phi + r^2$

$dx^2 + dy^2 + dz^2$

$= (dr)^2\sin^2\theta\cos^2\phi + r\,dr\cos^2\phi\sin\theta\cos\theta\, d\theta + r\,dr\sin\theta\cos\theta\, d\theta\cos^2\phi$
$+ r^2\cos^2\theta(d\theta)^2\cos^2\phi + r^2\sin^2\theta\sin^2\phi(d\phi)^2$] x

$+ (dr)^2\sin^2\theta\sin^2\phi + r\,dr\sin\theta\cos\theta\, d\theta\sin^2\phi + r\,dr\cos\theta\sin\theta\, d\theta\sin^2\phi$
$+ r^2\cos^2\theta(d\theta)^2\sin^2\phi + r^2\sin^2\theta\cos^2\phi(d\phi)^2$] y

$+ dr^2\cos^2\theta - r\,dr\cos\theta\sin\theta\, d\theta - r\,dr\sin\theta\cos\theta\, d\theta + r^2\sin^2\theta\, d\theta^2$

$a^2 - b^2 = (a-b)(a+b)$

$dz = dr\cos\theta - r\sin\theta\, d\theta$

$(dz)^2 = [\;"\;]^2 = dr\cos\theta\,(dr\cos\theta - r\sin\theta\, d\theta) - r\sin\theta\, d\theta\,(dr\cos\theta - r\sin\theta\, d\theta)$

$= dr^2\cos^2\theta - dr\cos\theta\, r\sin\theta\, d\theta - r\,dr\sin\theta\, d\theta\cos\theta + r^2\sin^2\theta\, d\theta^2$

$-2r\sin\theta\cos\theta\, dr\, d\theta$

$dx = dr\sin\theta\cos\phi + r(\cos\theta\, d\theta)\cos\phi - r\sin\theta(\sin\phi\, d\phi)$

$(dx)^2 = [\;"\;]^2$

$= dr\sin\theta\cos\phi\,[dr\sin\theta\cos\phi + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi]$

$(dr)^2\sin^2\theta\cos^2\phi + r\,dr\cos^2\phi\sin\theta\cos\theta\, d\theta - r\,dr\sin^2\theta\sin\phi\cos\phi\, d\phi$

$+ r(\cos\theta)\cos\phi\, d\theta\,[dr\sin\theta\cos\phi + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi]$

$+ r\,dr\sin\theta\cos\theta\, d\theta\cos^2\phi + r^2\cos^2\theta(d\theta)^2\cos^2\phi - r^2\cos\theta\sin\theta\, d\theta\sin\phi\cos\phi\, d\phi$

$- r\sin\theta\sin\phi\, d\phi\,[dr\sin\theta\cos\phi + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi]$

$- r\,dr\sin^2\theta\sin\phi\cos\phi\, d\phi - r^2\cos\theta\sin\theta\, d\theta\sin\phi\cos\phi\, d\phi + r^2\sin^2\theta\sin^2\phi(d\phi)^2$

$(dy)^2 = [\;"\;]^2$; $dy = dr\sin\theta\sin\phi + r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi$

$\hookrightarrow = dr\sin\theta\sin\phi\,[dr\sin\theta\sin\phi + r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi]$

$(dr)^2\sin^2\theta\sin^2\phi + r\,dr\sin\theta\cos\theta\, d\theta\sin^2\phi + r\,dr\sin^2\theta\sin\phi\cos\phi\, d\phi$

$+ r\cos\theta\, d\theta\sin\phi\,[dr\sin\theta\sin\phi + r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi]$

$+ r\,dr\cos\theta\sin\theta\, d\theta\sin^2\phi + r^2\cos^2\theta(d\theta)^2\sin^2\phi + r^2\sin\theta\cos\theta\, d\theta\sin\phi\cos\phi\, d\phi$

$+ r\sin\theta\cos\phi\, d\phi\,[dr\sin\theta\sin\phi + r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi]$

$+ r\,dr\sin^2\theta\cos\phi\sin\phi\, d\phi + r^2\cos\theta\sin\theta\, d\theta\sin\phi\cos\phi\, d\phi + r^2\sin^2\theta\cos^2\phi(d\phi)^2$

*yellow & blue cross out

$$(dr)^2\sin^2\theta\,\underbrace{[\cos^2\phi+\sin^2\phi]}_{=1} + r\,dr\sin\theta\cos\theta\,d\theta\,\underbrace{[\cos^2\phi+\sin^2\phi]}_{=1}$$

$$+\ r\,dr\sin\theta\cos\theta\,d\theta\,\underbrace{[\cos^2\phi+\sin^2\phi]}_{=1} + r^2\cos^2\theta(d\theta)^2\,\underbrace{[\cos^2\phi+\sin^2\phi]}_{=1}$$

$$+\ r^2\sin^2\theta(d\phi)^2\,\underbrace{[\sin^2\phi+\cos^2\phi]}_{\to 1}$$

$$+\ dr^2\cos^2\theta - r\,dr\cos\theta\sin\theta\,d\theta - r\,dr\sin\theta\cos\theta\,d\theta + r^2\sin^2\theta(d\theta)^2$$

z-component

+ pink cross out

$$\Rightarrow dr^2\underbrace{(\sin^2\theta+\cos^2\theta)}_{=1} + r^2(d\theta)^2\underbrace{[\cos^2\theta+\sin^2\theta]}_{\to 1}$$

$$+\ r^2\sin^2\theta\,d\phi^2$$

$$\therefore\ ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$\Rightarrow ds^2 = a^2(t)\underbrace{\left[(dx)^2 + (dy)^2 + (dz)^2\right]}_{\text{flat space}}$$

$$\Rightarrow ds^2 = -c^2dt^2 + (a(t))^2 \underbrace{\left[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]}_{\text{spherical coordinates}}$$

$$\Rightarrow ds^2 = -c^2dt^2 + [a(t)]^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

Impose cosmological principal — universe has no preffered locations — means that curvature must be constant.

Above is the "flat case". However lets look at all possible curvatures

Let's look at distance between points on a sphere.

$x^2 + y^2 + z^2 = R^2$ <equation of sphere>

$r^2 + z^2 = R^2$ <switch to polar coordinates, so $x^2 + y^2 = r^2$>

We also have Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$

Converted to polar coordinates $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

Now take metric and apply it onto the sphere:

$zdz = -rdr$ ← if we move on the surface of the sphere, this tells us how much dr and dz are changing to one another.

$$ds^2 = dr^2 + r^2 d\theta^2 + \frac{r^2 dr^2}{z^2}$$

← $zdz = -rdr$

$dz = \frac{-rdr}{z} \Rightarrow (dz)^2 = \frac{r^2 dr^2}{z^2}$

(plugged in for dz)

$$\Rightarrow ds^2 = \left(1 + \frac{r^2}{z^2}\right)dr^2 + r^2 d\theta^2$$

Using $r^2 + z^2 = R^2$ we may rewrite it as $z^2 = R^2 - r^2$
plug this into our equation:

$$ds^2 = \left(1 + \frac{r^2}{z^2}\right) dr^2 + r^2 d\theta^2 = \left(1 + \frac{r^2}{R^2 - r^2}\right) dr^2 + r^2 d\theta^2$$

also: $1 + \frac{r^2}{R^2 - r^2} = \frac{R^2 - r^2}{R^2 - r^2} + \frac{r^2}{R^2 - r^2} = \frac{R^2}{R^2 - r^2}$

$$\Rightarrow \frac{R^2 (1/R^2)}{(R^2 - r^2)(1/R^2)} = \frac{1}{1 - \frac{r^2}{R^2}}$$

Full form: $ds^2 = \left(\frac{dr^2}{1 - \left(\frac{r}{R}\right)^2}\right) + r^2 d\theta^2$

In spherical coordinates it becomes:

$$ds^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r d\theta)^2 + (r \sin\theta \, d\phi)^2$$

* On a surface of a sphere, curvature is defined as $K \equiv \frac{1}{R^2}$

Now we must incorporate time dependances:

$$\Rightarrow ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

This is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric.

k can take up 3 possible values:

$k = +1$ spherical geometry $\Rightarrow ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + \sin^2 r\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right.$

$k = 0$ flat geometry $\Rightarrow ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$

$k = -1$ hyperbolic geometry $\Rightarrow ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + \sinh^2 r\,(d\theta^2 + \sin^2\theta\, d\phi^2\right.$

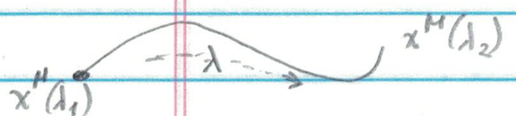
This metric describes an isotropic universe because it does not have crossed terms between time and space thus there is no particular direction either. It's spherical symmetry guarantees a homogeneous universe.

Calculating Christoffel Symbols for FLRW metric

formula for Christofell symbols : $\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right]$

The geodesic is specifically the shortest path of a particle in the absence of any external lines. It is a more elaborate idea of a "straight line"

The geodesic equation is : $\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$



The christofell symbols are the coefficients that are constructed from the metric and its first derivatives.

The results will be :

t: $\Gamma^{t}_{rr} = \frac{a\dot{a}}{1-\frac{r^2}{K^2}}$ $\quad \Gamma^{t}_{\theta\theta} = r^2 a\dot{a}$ $\quad \Gamma^{t}_{\phi\phi} = r^2 a\dot{a}\sin^2\theta$

r: $\Gamma^{r}_{tr} = \Gamma^{r}_{rt} = \frac{\dot{a}}{a}$ $\quad \Gamma^{r}_{rr} = \frac{r}{K^2(1-\frac{r^2}{K^2})}$

$\Gamma^{r}_{\theta\theta} = -r\left(1-\frac{r^2}{K^2}\right)$ $\quad \Gamma^{r}_{\phi\phi} = -r\left(1-\frac{r^2}{K^2}\right)\sin^2\theta$

θ: $\Gamma^{\theta}_{t\theta} = \Gamma^{\theta}_{\theta t} = \frac{\dot{a}}{a}$ $\quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}$ $\quad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$

ϕ: $\Gamma^{\phi}_{t\phi} = \Gamma^{\phi}_{\phi t} = \frac{\dot{a}}{a}$ $\quad \Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}$ $\quad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{1}{\tan\theta}$

all others $\longrightarrow 0$.

$$ds^2 = dt^2 - a^2(t)\left[\frac{1}{1-\frac{r^2}{R^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]$$

$$\mathcal{L} = \frac{1}{2} m \left(\frac{ds}{d\tau}\right)^2 \qquad \dot{t} = \frac{dt}{d\tau}$$

$$\Rightarrow \frac{1}{2} m \left[\dot{t}^2 - a^2(t)\left[\frac{1}{1-\frac{r^2}{R^2}} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right]\right]$$

θ - coordinate - example -

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \left(\frac{1}{2} m\right)\left(a^2(t)\, r^2 (2\dot{\theta})\right) = m r^2 a^2(t)\, \dot{\theta}$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m (2 r \dot{r})\, \dot{\theta}\, a^2(t) + m r^2 a^2(t)\, \ddot{\theta} + m r^2 \dot{\theta}\, 2 a(t)\, \dot{a}(t)\, \dot{t}$$

$$\dot{\theta} \neq \theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{2} m\, a^2(t) \left(r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right)\right)$$

$$= \frac{1}{2} m\, a^2(t)\left(r^2 (2 \sin\theta \cos\theta)\, \dot{\phi}^2\right) = m\, a^2(t)\, r^2 \sin\theta \cos\theta\, \dot{\phi}^2$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$0 = 2 m r\, a^2(t)\, \dot{r} \dot{\theta} + m r^2 a^2(t)\, \ddot{\theta} + 2 m r^2 a(t)\, \dot{\theta}\, \dot{a}(t) - r^2 m\, a^2(t) \sin\theta \cos\theta\, \dot{\phi}^2$$

solve for $\ddot{\theta}$

$$\frac{m r^2 a^2(t)\, \ddot{\theta}}{m r^2 a^2(t)} = \frac{2 m r a^2(t)\, \dot{r} \dot{\theta}}{m r^2 a^2(t)} + \frac{2 m r^2 a(t)\, \dot{\theta}\, \dot{a}(t)}{m r^2 a^2(t)} - \frac{r^2 m a^2(t) \sin\theta \cos\theta\, \dot{\phi}^2}{m r^2 a^2(t)}$$

$$\ddot{\theta} = \frac{2}{r} \dot{r} \dot{\theta} + \frac{\dot{a}(t)}{a(t)} \dot{\theta} \dot{t} - \sin\theta \cos\theta\, \dot{\phi}\dot{\phi}$$

so $\Gamma^{\theta}_{r\theta} = \frac{2}{r}$; $\Gamma^{\theta}_{\theta t} = \frac{\dot{a}(t)}{a(t)}$; $\Gamma^{\theta}_{\phi\phi} = -\sin\theta \cos\theta$

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