Date of publication xxxx 00, 0000, date of current version xxxx 00, 0000.

Digital Object Identifier 10.1109/ACCESS.2024.0429000

Harmonic Oscillator based Particle Swarm Optimization

YURY CHERNYAK[1], IJAZ AHAMED MOHAMMAD[1], Nikolas Masnicak[1], Matej Pivoluska[1,2] and Martin Plesch[1,3]
[1]Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 5807/9, 845 11 Karlova Ves, Bratislava, Slovakia
[2]QTlabs, Clemens-Holzmeister-Straße 6/6 Etage 6, 1100 Wien, Austria
[3]Matej Bel University, Národná ulica 12, 974 01 Banská Bystrica, Slovakia

Corresponding author: Ijaz Ahamed Mohammad (e-mail: fyziijaz@savba.sk).

We acknowledge the support of VEGA project 2/0055/23 and projects 09I03-03-V04-00425 and 09I03-03-V04-00685 of the Research and Inovation Authority.

ABSTRACT Numerical optimization techniques are widely used in a broad area of science and technology, from finding the minimal energy of systems in Physics or Chemistry to finding optimal routes in logistics or optimal strategies for high speed trading. In general, a set of parameters (parameter space) is tuned to find the lowest value of a function depending on these parameters (cost function). In most cases the parameter space is too big to be completely searched and the most efficient techniques combine stochastic elements (randomness included in the starting setting and decision making during the optimization process) with well designed deterministic process. Thus there is nothing like a universal best optimization method; rather than that, different methods and their settings are more or less efficient in different contexts. Here we present a method that integrates Particle Swarm Optimization (PSO), a highly effective and successful algorithm inspired by the collective behavior of a flock of birds searching for food, with the principles of Harmonic Oscillators. This physics-based approach introduces the concept of energy, enabling a smoother and a more controlled convergence throughout the optimization process. We test our method on a standard set of test functions and show that in most cases it can outperform its natural competitors including the original PSO as well as the broadly used COBYLA and Differential Evolution optimization methods.

INDEX TERMS Convergence, Global Optimization, Local Minima, Meta-heuristic optimization, Multi-modal problems, Optimization algorithms, Particle swarm optimization, Swarm Intelligence

I. INTRODUCTION

Meta-heuristic optimization techniques are a popular way to perform unconstrained minimization of complicated functions. These techniques are often inspired by natural phenomena, animal behaviors, or evolutionary concepts, making them easy to learn, implement, and hybridize. In addition, they are flexible and can be applied to various problems without altering their structure, treating problems as black boxes where only inputs and outputs matter. Unlike gradient-based approaches, meta-heuristics optimize stochastically without needing derivatives, making them suitable for complex problems for which derivatives are hard to obtain. Finally, their stochastic nature also helps avoid local optima, making them effective for challenging real-world problems with complex search spaces.

Despite quite a large number of different meta-heuristic methods published over the years, there is still an ongoing research in this area with current approaches being enhanced and new meta-heuristics being proposed frequently. This is due to the No Free Lunch (NFL) theorem [1], which states that no single meta-heuristic is best for all optimization problems. An algorithm might perform well on one set of problems but poorly on another. This drives ongoing improvements and the development of new meta-heuristics, motivating our efforts to create a new one.

Meta-heuristics can be classified into single-solution-based and population-based methods. Single-solution methods, like Simulated Annealing [2], start with one candidate solution that is improved iteratively. In contrast, population-based methods, like Particle Swarm Optimization [3] (PSO), begin with multiple solutions that are enhanced over iterations. Advantages of population-based methods include information sharing among solutions, which leads to sudden jumps to promising areas, mutual assistance in avoiding local optima, and generally greater exploration compared to single-solution algorithms.

Population based meta-heuristics algorithms can further be classified into three main branches: evolutionary, physics-based, and swarm intelligence algorithms. Evolutionary algorithms, inspired by natural evolution, optimize by evolving an initial population of random solutions. The most popular algorithm in this class is the Genetic Algorithm (GA) [4], simulating Darwinian concepts. Each new population is formed by combining and mutating individuals from the previous generation, with the best individuals more likely to contribute, ensuring gradual improvement. Other evolutionary algorithms include Differential Evolution (DE) [5], Evolutionary Programming (EP) [6], Evolution Strategy (ES) [7], Genetic Programming (GP) [8], and Biogeography-Based Optimizer (BBO) [9].

The second main branch of meta-heuristics is physics-based techniques, which mimic physical rules. Popular algorithms include Gravitational Local Search Algorithm (GLSA) [10], Big-Bang Big-Crunch (BBBC) [11], Gravitational Search Algorithm (GSA) [12], Charged System Search (CSS) [13], Central Force Optimization (CFO) [14], Artificial Chemical Reaction Optimization Algorithm (ACROA) [15], Black Hole (BH) algorithm [16], Ray Optimization (RO) algorithm [17], Small-World Optimization Algorithm (SWOA) [18], Galaxy-based Search Algorithm (GbSA) [19], and Curved Space Optimization (CSO) [20]. These algorithms use a random set of search agents that move and communicate according to physical rules, such as gravitational force, ray casting, electromagnetic force, and inertia.

The third subclass of meta-heuristics is Swarm Intelligence (SI) methods, which mimic the social behavior of groups in nature. Similar to physics-based algorithms, these use search agents navigating through collective intelligence. The most popular SI technique is Particle Swarm Optimization (PSO), proposed by Kennedy and Eberhart [3], inspired by bird flocking behavior. PSO employs multiple particles that move based on their own best positions and the best position found by the swarm. Other algorithms in this class are Ant Colony Optimization (ACO) [21], Artificial Bee Colony (ABC) [22], Bat-inspired Algorithm (BA) [23], Marriage in Honey Bees Optimization Algorithm (MHBO) [24], Artificial Fish-Swarm Algorithm (AFSA) [25], Termite Algorithm (TA) [26], Wasp Swarm Algorithm (WSA) [27], Monkey Search (MS) [28], Bee Collecting Pollen Algorithm (BCPA) [29], Cuckoo Search (CS) [30], Dolphin Partner Optimization (DPO) [31], Firefly Algorithm (FA) [32], Bird Mating Optimizer (BMO) [33], Krill Herd (KH) [34], Fruit fly Optimization Algorithm (FOA) [35] and Grey Wolf Optimizer [36].

Among this pool of methods, Particle Swarm Optimization (PSO) stands out as a significant and relevant algorithm for several reasons such as its simplicity and ease of implementation, as well as outstanding results in test cases and real deployment. Unlike Genetic Algorithms (GA) and Ant Colony Optimization (ACO), PSO requires fewer parameters to adjust, resulting in a reduced computational burden [37], [38], [39]. PSO also benefits from its internal "memory" capabilities; it leverages previously known best positions to enhance search efficiency, unlike GAs. Its scalability is another key feature, as PSO can effectively handle large-scale, high-dimensional complex optimization problems. In terms of global optimization, PSO demonstrates the ability to avoid multiple local minima, allowing it to navigate rugged landscapes and identify the global minimum. Additionally, PSO's flexibility enables it to be easily hybridized with other optimization techniques, further improving its results [40], [41], [42].

As one could expect, despite the numerous benefits mentioned above, PSO has a few drawbacks to show as well. One of the major issues it faces is the occasional uncontrolled movement of some of the birds in the flock, which can lead to poor convergence. More specifically, even if the optimal points determined by the birds so far are confined to a small region, the birds can gain very high velocities, resulting in an expansion of the search space into irrelevant, large regions. On other occasions, birds lose their velocity very quickly and then proceed to get stuck on a point despite the optimization needing to continue.

A natural approach to address these issues is to get under control the reduction as well as the possible increase of the velocity of the birds. As we show in the next section, however, this is not that easy, and for this purpose, the commonly used settings for the hyper-parameters of PSO are not always practical. This is why we introduce, in Section III, a new population-based meta-heuristic method, that introduces the concept of energy (a combination of velocity and distance from the optimal positions known thus far) into PSO. This allows to control the convergence of the method independently of other parameters, while still keeping all the existing PSO features intact. While the main design inspiration is based on the PSO algorithm, this method is better classified as a physics based meta-heuristic, because the movement of the particle population is governed by the physics of harmonic oscillators – hence we name the method as Harmonic Oscilator based Particle Swarm Optimization (HOPSO).

Results of the HOPSO method are described in Section IV, showcasing its capabilities on 12 different test functions, while the Section V concludes our findings.

II. INTRODUCTION TO PSO AND ITS PROBLEMS

The Particle Swarm Optimization (PSO) is a meta-heuristic population based optimization method that works by simulating the social behavior of a flock of birds or a school of fish. In these social systems, the movement of the swarm species was observed to be a form of optimization in their search for food.

Modeling after this swarm behavior, the PSO was developed such that each particle represents a potential solution, and it modifies its position in the search space according to the individual experience (a "cognitive" term) and the group experience (a "social" term).

PSO is simple and easily implementable, and it can ensure fast convergence to a satisfactory solution with a small number of control parameters. It works well even when the search space is large or when the problem is highly non-linear or multi-modal.

PSO also frequently outperforms other algorithms on problems with a smooth landscape, where the optimization process would benefit from a more exploratory approach that can be realized by the collective behavior of the swarm – this is particularly important in the newly developing field of Variational Quantum Algorithms [43], [44].

To briefly describe the working of PSO, the algorithm begins with a population of N particles, each representing a candidate solution in the d-dimensional search space of the optimization problem under consideration. The positions of these particles are randomly initialized within predefined search space boundaries and move according to specific update equations in discrete time steps, i.e. iterations. Specifically, the determination of the position in the subsequent iteration is dependent on the current position with an additional velocity-vector that drives the particle to a new, and ideally better, position in the search space. This velocity vector for the next iteration incorporates three crucial components: the inertia (previous velocity), the cognitive component, and the social component. The cognitive component represents the element-wise difference between the personal best position vector and its current position vector, while the social component represents the element-wise difference between the global (swarm) best position vector and the particle's current position. Each of these is then scaled by the 'cognitive' and 'social' coefficients, denoted as c_1 and c_2, respectively. These coefficients represent the relative importance assigned to the particle's personal best position (stored in its individual memory) and the swarm's global best position when calculating the velocity for the next iteration, thereby determining the relative importance of the particle's personal experience versus the swarm's collective knowledge in calculating the next velocity and position.

Moreover, each of these terms is adjusted by a unique random factor, thereby introducing stochasticity into the search process. This movement is illustrated in figure 1. To allow convergence of the whole system, a damping factor in the form of a *constrictor factor* χ was introduced in [45]. The update equations of this variation of the PSO algorithm are introduced as follows with a description of its variables in Table 1.

Velocity Update:

$$v_{j,d}(i+1) = \chi(v_{j,d}(i) + c_1 r_1 (p_{j,d} - x_{j,d}(i)) + c_2 r_2 (g_d - x_{j,d}(i))) \tag{1}$$

Position Update:

$$x_{j,d}(i+1) = x_{j,d}(i) + v_{j,d}(i+1). \tag{2}$$

Balancing exploration capabilities and reasonable convergence is the main challenge across all PSO variants. A low constriction factor causes particles to stop moving too quickly, while high values (leading to low damping) cause particles to spread into large regions far from any optima. A specific form of the constrictor factor, χ, has therefore been derived from the stability analysis of the PSO system to address these concerns regarding the values of the velocity term. Particularly, the velocity update equation can be viewed as a second-order difference equation which occurs when removing random variables from the PSO equation and is then examined using roots of its characteristic equation. The roots, λ, determine the behavior of the velocity over time and thereby, for the system to be stable (converge), the absolute value of the roots must be less than 1. Thus, the form of χ must be such that the eigenvalues of the system are indeed less than 1. Also, if the parameters c_1 and c_2 are too large, the system also becomes unstable, leading to divergence of the particle velocities. Their sum $\varphi = c_1 + c_2$ directly impacts the constrictor factor χ and, consequently, the magnitude of the velocity update. Larger values of φ causes the constrictor factor to dampen the velocities more significantly to prevent runaway velocities that lead to divergence, therefore allowing a better and more controlled convergence behavior. Lastly, the presence of the square root and absolute value in χ ensures they reduce gradually as they approach an optimal solution, preventing overshooting or oscillatory behavior. From these observations in their analysis, Clerc and Kennedy proposed the mathematically designed form of χ to be

$$\chi = \frac{2}{\left|2 - \varphi - \sqrt{\varphi^2 - 4\varphi}\right|} \tag{3}$$

$$\varphi = c_1 + c_2. \tag{4}$$

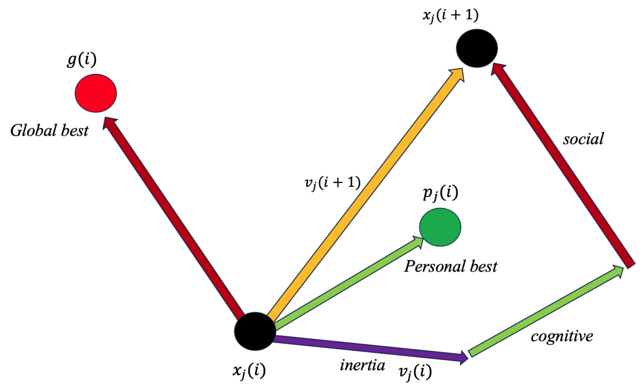

FIGURE 1: An example of the movement of a particle in a two-dimensional space based on the PSO algorithm in a single iteration. An inertia term given by velocity that drives the particle in some direction (violet arrow), a memory term (p_j) that influences the particle's trajectory based on its best known position (green arrow), and a global-best cooperation term (g) that reflects the best result amongst the entire swarm (red) constitute the particle's projected movement (yellow arrow). The i indicates iteration, while j indicates particle number.

Empirical studies suggest that commonly used parameters to ensure convergence are $c_1 = c_2 = 2.05$ [46], with a constriction factor $\chi = 0.7298$, derived from the analysis of PSO without randomness.

TABLE 1: Explanation of PSO parameters

Parameter	Description
$v_{j,d}(i)$	Velocity of particle j in dimension d at iteration i
$x_{j,d}(i)$	Position of particle j in dimension d at iteration i
χ	Constrictor factor (damping)
c_1	Cognitive coefficient, attraction towards the particle's best known position
c_2	Social coefficient, attraction towards the swarm's best known position
r_1, r_2	Random values uniformly distributed in the range $[0, 1]$
$p_{j,d}$	d-th dimension of the best known position of particle j
g_d	d-th dimension of the global best known position

Let us now analyse in detail the potential for velocity explosions in PSO, a phenomenon described by Kennedy and Eberhart [45]. This occurs when particle velocities significantly exceed the characteristic scale of the search space during the optimization process, leading to swarm divergence, as these high-velocity particles continue to traverse the parameter space rapidly and their ability to effectively locate optima of the cost function becomes severely compromised. This behavior thereby undermines the balance between exploration and exploitation that is crucial for the PSO's effectiveness.

In order to understand why these velocity explosions occur, let us briefly analyse the time evolution of a single bird in our swarm. For simplicity, let us consider only one dimension, as each of the dimensions are behaving independently until a new optimum is identified. In this case, the position update equation (2) and velocity update equation (1) can be represented with a matrix M acting on the vector with position and velocity coordinates which thereby provides a compact way to describe the coupled dynamics of both velocity and position written as presented in (5)

$$\vec{P}_{t+1} = M\vec{P}_t, \quad \vec{P}_t = (v_t, y_t), \tag{5}$$

where

$$y_t = \frac{\varphi_1 g + \varphi_2 p}{\varphi_1 + \varphi_2} - x_t, \tag{6}$$

with $\varphi_1 = r_1 c_1$ and $\varphi_2 = c_2 r_2$ and the dynamical matrix M governing the time evolution defined as

$$M = \begin{pmatrix} \chi & \chi\varphi \\ -\chi & 1-\chi\varphi \end{pmatrix}, \tag{7}$$

where $\varphi = \varphi_1 + \varphi_2$.

Notice that equation (5) only represents a single iteration step but since the random numbers r_1 and r_2 are changing in each iteration, to represent the state of the particle after several iterations we need to accumulate the effects of all previous iterations, therefore the state vector $\vec{P}_t$ of a bird that was initially at position x_0 with velocity v_0 can be described as a product of transformations

$$\vec{P}_t = \prod_{i=0}^{t} M_i \vec{P}_0, \quad \vec{P}_0 = (v_0, y_0). \tag{8}$$

If no new local or global optimum is found, it is expected that our bird will gradually converge and its velocity will decay to zero. As analysed in [45], when one keeps the same value of χ and φ throughout the entire simulation (removing the stochastic nature of the method), the sufficient condition for convergence is

$$\max(|\lambda_1|, |\lambda_2|) < 1, \tag{9}$$

where λ_1 and λ_2 are the eigenvalues of M given by

$$\lambda_{1,2} = \frac{1}{2}\left(1 + (1-\varphi)\chi \pm \sqrt{((\varphi-1)\chi - 1)^2 - 4\chi}\right), \tag{10}$$

as both the eigenvectors are multiplied by λ_1^t and λ_2^t which both converge to 0.

This is no longer the case when one includes the randomness into PSO. In this case the behavior is governed by a product of t different dynamical matrices in (8). Here, despite the fact that each individual matrix M has the magnitude of both eigenvalues below 1, it is in general not the case for their product, i.e. subsequent multiplication of matrices, with each of their eigenvalue being smaller than one, to also form a matrix with a very small eigenvalue, therefore having large eigenvalues will lead to explosions in velocities.

To analyse the product of different matrices, one has rather to look into the singular values of the dynamical matrix M given by

$$\sigma_{1,2}^2 = \frac{1}{2}\Big(2\chi^2\big((c_1r_1 + c_2r_2)^2 + 1\big) - 2(c_1r_1 + c_2r_2)\chi + 1 \\ \pm\sqrt{(2\chi^2\big((c_1r_1 + c_2r_2)\big)^2 + 1) - 2(c_1r_1 + c_2r_2)\chi + 1)^2 - 4\chi^2}\Big). \tag{11}$$

Singular values express the limits of maximum possible prolongation or contraction of a vector that is multiplied by a given matrix. To have the length of the vector under control, one optimally needs to have both singular values being similar, and for smooth convergence it must be smaller than one but close to one.

Using a simplification of $c_1 = c_2 = 2.05 = c$ along with definition of $r = r_1 + r_2$ as the sum of random numbers used in the specific step, expression (12) becomes

$$\sigma_{1,2}^2 = \frac{1}{2}\Big(2\chi^2\big((cr)^2 + 1\big) - 2cr\chi + 1 \\ \pm\sqrt{\big(2\chi^2(cr)^2 + 1) - 2cr\chi + 1\big)^2 - 4\chi^2}\Big). \tag{12}$$

Singular values for matrix (7) are shown in Fig. 2 – note that the sum of random numbers runs from 0 to 2.

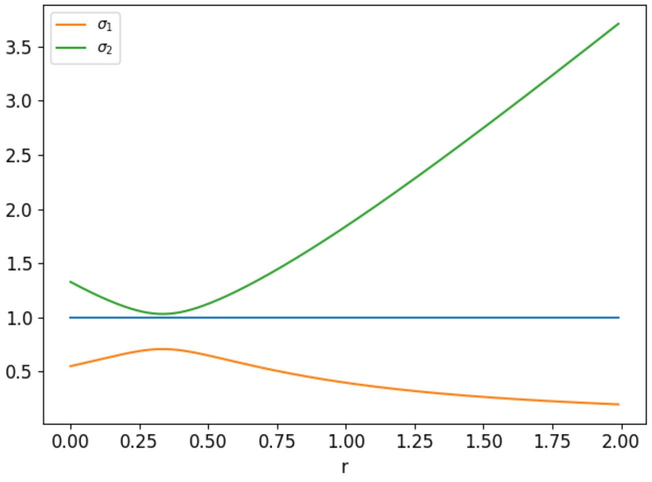

FIGURE 2: Singular values of dynamical matrix M for $c_1 = c_2 = 2.05$, in which case $\chi = 0.729$. One can see that in the region of small sum of random numbers both singular values are close to one and their average is below one, so one can expect some kind of reasonably small convergence. However, for large sum of random numbers this is not the case – one singular values almost diminishes and the other one reaches almost the value of 4, thus one can, depending on the starting position and exact combination of random numbers obtain both very fast dying out, as well as velocity and position explosions. Let us note, however, that using two independent random numbers r_1 and r_2 makes the probability distribution of its sum higher, when the sum of r_1 and r_2 is around 1, making the explosions less probable.

Results of this theoretical analysis have been confirmed by having studied the behavior of the method in specific real-world cases. First of all, the explosions have been seen as resulting from a specific series of random numbers, where large and small numbers have regularly followed after each other. This is because the specific form of M matrix causes rotation in the governing vector (large velocity in one step induces large distance in the next one), leading to the application of the second singular value in the next step. If, however, a large random number is applied when the larger singular value is active, along with a small random number combined with smaller singular value, the resulting behavior is divergent.

This understanding is also confirmed by the observed fact, that when one decided to use only a single random number (i.e. defining $r_1 = r_2$), the velocity explosions happened regularly, while other birds died very quickly. This is due to the fact that the extreme singular values were achieved with a much higher probability compared to when there were two independent random numbers.

It is clear that achieving a smooth and controlled convergence of the method with its existing governing equation, a very detailed control of randomness applied would be necessary. This would, however, compromise the stochastic nature of the method – the more restrictions that are applied, the higher is the probability of not reaching the global maximum. Thus, we suggest to re-formulate the PSO idea in a more physical framework, introducing the concept of energy, its conservation and loss via damping, together with possible energy boosts connected with finding a new global or local optimum.

Achieving smooth and controlled convergence with the current governing equations would require precise management of the randomness applied. However, this would undermine the stochastic nature of the method—the more constraints imposed, the higher the likelihood of failing to reach the global maximum. Therefore, we propose reformulating the PSO approach within a more physical framework, introducing concepts such as energy, conservation, and loss through damping, along with potential energy boosts associated with finding new global or local optima.

III. HARMONIC OSCILLATOR BASED PARTICLE SWARM OPTIMIZATION – HOPSO

The general framework of energy consists of two basic types – kinetic energy, defined by the movement of specific particles, and potential energy, defined in most cases by the position of particles and expressing the potential to gain (or loose) kinetic energy by changing its position. Potential energy can only be associated by conservative forces, such as gravity or electromagnetic forces, whereas friction forces are typical examples of non-conservative forces that lead to loss of energy (or, more precisely, to dissipation into heat).

While there exist optimization approaches that are based on gravity [12], [47], which is mathematically very similar to the the model of charges, we consider it as not a very promising approach. Firstly, the energy diverges to negative infinity while reaching the center of gravity, which leads, again, to explosions of velocities. Secondly, for long distances the attractive force only increases in a minor way, allowing the particles to fly very far from the attractor.

For such reasons, we were inspired to use spring forces, e.g. forces that are linear with the distance between the attractor and the position of the particle, and can be associated with an ideal spring. This concept is optimal as for small distances the potential energy converges to 0 while for large distances the energy grows quadratically, essentially bounding the particle to a well defined region.

One challenge is that when a single spring is used across multiple dimensions, the total energy depends in a complex non-trivial way on the position in all dimensions. To simplify the process, we propose a model in which an independent virtual spring is assigned to each dimension, attracting the particle within that dimension only. This approach maintains the principle of energy conservation while enabling faster and simpler calculations. It also allows for independent adjustment of constraints in different dimensions, as the energy is now decoupled for each parameter.

Now the movement of particles, if modelled in continuous time, would reflect swinging on a spring with a center defined by an attractor (a suitable combination of social and cognitive term) independently in each single dimension. The period of these oscillations is defined by a combination of a virtual mass of that particle and the stiffness of the spring – each of which can be chosen arbitrarily. This movement has a simple analytical solution – harmonic motion.

In the optimization process, there is no need to model the entire motion. Instead, snapshots are taken at different moments during the harmonic oscillations. Randomness starts to influence the process in HOPSO at this stage. Rather than altering the potential (by adjusting the constants that link position and velocity changes) which would cause unpredictable energy fluctuations, we instead allow the particles to oscillate harmonically and observe them as snapshots at different time intervals.

In contrast to the original PSO where the damping influences multiple aspects including the social and cognitive terms through the constrictor factor, here the damping constant solely governs energy dissipation which directly relates to the searching ability of the particle. In the results section we will show how to use this parameter, indicating that not only does this parameter offer more tunability in governing convergence via this physically-inspired model, but also that this damping parameter is an easily controlled independent parameter (i.e. can be classified as a "free parameter"). This flexibility offers a significant advantage for optimization problems, and is one of the crucial advantages of the HOPSO algorithm over not only the standard PSO method, but other non-gradient methods as well, presented in our results section.

More formally, each particle's position in each dimension is defined as the solution of a damped harmonic oscillator. The solution is given by the following equation:

$$x(t) = A_0 e^{-\lambda t} \cos(\omega t + \theta) + x_0, \tag{13}$$

where, $x(t)$, A_0, λ, ω and θ represent the position of the particle at time t, initial amplitude, damping factor, angular frequency and initial phase of the oscillation, respectively, whereas the x_0 is the position of the attractor in that direction.

The velocity of the particle at any given time t can be obtained by differentiating (13)

$$v(t) = -\omega(A_0 e^{-\lambda t} \sin(\omega t + \theta)) - \lambda(x(t)). \tag{14}$$

The time of measurement is chosen randomly within the interval $[0, t_{ul}]$ for each particle and in each dimension, where t_{ul} is the upper limit of the time sampling range and typically is chosen as the period of oscillation. The iterative change in parameter t is defined as (15)

$$t_{i+1} = t_i + rand[0, t_{ul}], \tag{15}$$

where the parameter i represents the index of iteration.

The optimization begins by initializing the particles at random positions along with random velocities in the parameter landscape. The initial positions are noted as the initial personal-best positions. The global-best position is noted as the best position out of all the personal-best positions of all the particles. This global best position corresponds to the lowest value of the cost function within the swarm.

Each particle then oscillates about an attractor independently in each dimension. This attractor can be calculated as

$$a_{j,d} = \frac{c_1 p_{j,d} + c_2 g_d}{c_1 + c_2}, \tag{16}$$

where a, p, g represent, respectively, the position of the attractor, personal best position of the particle and the global best position for the j^{th} particle in d^{th} dimension. The c_1 and c_2 terms represent the weights of attraction towards the personal-best and global-best positions, respectively. Typically, the values c_1 and c_2 are set equal, so the attractor lies equidistant between the personal best position and the global best position.

We solve (13) and (14) to obtain the initial amplitudes A_0 for each particle and in each dimension by choosing the initial time as $t = 0$:

$$A_0 = \sqrt{(x(0) - a)^2 + \frac{(v(0) + \lambda(x(0) - a))^2}{\omega^2}}. \tag{17}$$

Once the initial oscillation amplitude A_0 is determined, the initial phase of the oscillation θ can be calculated as

$$\theta = \arccos \frac{x(0) - a_{j,d}}{A_0}. \tag{18}$$

We then let the particles oscillate in time, while this initial amplitude of oscillation A_0 decays as $A_0 e^{-\lambda t}$. For every iteration, we stop the clock at a random time, calculate the values of the cost function for all the particle positions.

If there is a change in p_j, then the attractor for the j^{th} particle is recalculated using (16) while the amplitudes and phase values are recalculated by resetting the time for that particular particle as zero using (17), (20) & (21). If instead there is a change in g, then all the attractors are changed accordingly using (16). The amplitudes and phases are again recalculated using (17), (20) & (21) by resetting the time as zero for all particles.

This procedure, however, might in some cases lead to a significant loss of energy for the particle that found the new best position. This can be seen in an example when the best position is found at the boundary of the oscillation region where the velocity of the particle is close to zero. If that position is the new attractor, the potential energy of the string will be zero as well, as it freezes the particle until a new global best position is found. This is naturally not a desirable situation, as that particle is "punished" for being successful. To deal with this issue, we postulate a condition that the newly calculated amplitude is never smaller than the previous one – more generally, finding a new best position can never lead to decrease of the energy in the system.

Given enough time without finding a new global best, there is a possibility that the amplitude of particles whose personal best is far away from the global best becomes so small that the particle fails to effectively search the space between them, causing it to virtually become stuck in the middle, which might be a very unfavorable region. Numerical simulations did show that this basically leads to an effective loss of this kind of particle, as it will almost never contribute with new results. A naive approach to solve this problem is by reducing the damping parameter λ, however the drawbacks of this is that this may also reduce the convergence in the desired cases.

We therefore included a different approach to resolve this issue, namely, to limit the amplitude from below to some threshold amplitude A_{th} and prevent it from further damping. Naturally, this treshold amplitute should be in the order of the distance between the global and personal best positions to allow for searching in a region encompassing both of them. Thus we define

$$A_{th} = \frac{|p_{j,d} - g_d|}{2} * m, \tag{19}$$

where m is a free parameter and define

$$(A_0)_{i+1} = \max((A)_i, (A_0)_{i+1}, A_{th}). \tag{20}$$

A pictorial representation of (20) is shown in the Fig. 3.

A pseudo-code formalizing all the previously described procedures is shown in Algorithm 1.

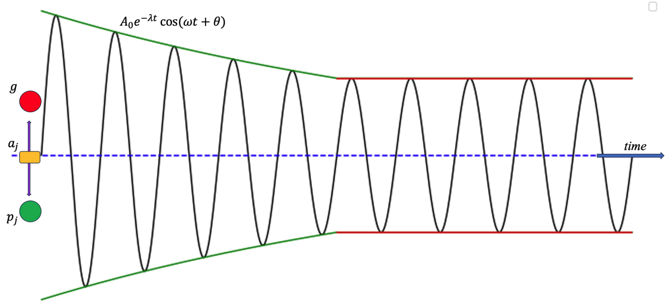

FIGURE 3: HOPSO Visualization: In one-dimension, the particle j oscillates about the attractor a_j which is set half-way between its personal best (p_j) and the swarm's global best (g) based on the weighted average equation (16). The damping is switched off when the amplitude decreases, in the depicted case, it is twice the distance between the attractor and one of the best positions.

Algorithm 1: Harmonic Oscillator based Particle Swarm Optimization (HOPSO)

```
Input: Problem dimensions, Objective function,
       Algorithm parameters
Output: Candidate Optimal solution
Set constants $c_1$, $c_2$, $\lambda$, $m$ for attraction weights and
  damping and minimal amplitude ration;
Initialize particles with random positions $x_{i,d}$ and
  velocities $v_{i,d}$;
Set initial personal best positions $p_i$ by starting
  positions for each particle;
Choose initial global best position $g$;
Calculate position of attractors: $a_i = \frac{c_1 p_i + c_2 g}{c_1 + c_2}$;
Calculate initial amplitude:
  $A_0 = \sqrt{(x(0) - a)^2 + \left(\frac{v(0)+\lambda(x(0)-a)}{\omega}\right)^2}$;
Calculate initial phase: $\theta = \arccos\left(\frac{x(0)-a}{A_0}\right)$;
while $iteration < max\_iterations$ do
    foreach particle do
        foreach dimension do
            $A = \max(A_0 e^{-\lambda t}, \frac{|p_i - g|}{2} \cdot m)$;
            $x(t) = A_0 e^{-\lambda t}\cos(\omega t + \theta) + x_0$;
            $v(t) = -\omega(A_0 e^{-\lambda t}\sin(\omega t + \theta)) - \lambda x(t)$;
    foreach particle do
        Calculate Cost function from positions;
        if $Cost\_function(x_{i,d}(t)) < Cost\_function(p_i)$
          then
            Update $p_i$, best value, time, attractors,
              amplitude, phase;
    if personal best value < global best energy then
        Update global best value and $g$;
        foreach particle and dimension do
            Reset time, recalculate attractors,
              amplitude, phase;
    iteration ← iteration + 1;
```

IV. RESULTS

We demonstrate now the performance of our algorithm on on a large set of commonly used test functions for multi-variable many-minima test functions. These functions are listed in Tab. 2.

These particular functions were chosen for their different and diverse properties, making them ideal for testing optimization algorithms [48] and are a common, standard choice. The Ackley and Rastrigin functions have many local minima. Similarly, the Levy function's complex landscape challenges algorithms to avoid local minima. The Sphere function is a simple, unimodal, bowl-shaped function, while the Beale function is multimodal with sharp peaks at the corners of the input domain. The Goldstein-Price function is highly multimodal and complex, while the Schwefel function, known for its large search space, presents numerous traps. The Rosenbrock function's narrow, curved valley tests precision, while the Drop-Wave function features steep drops and peaks. The Cross-in-Tray function and the Michalewicz function, with their deep valleys and sharp peaks, also make it extremely challenging for optimization algorithms to find the optimum and are therefore suitable choices as well. Each function was chosen to uniquely test different aspects of the optimization algorithm that we present in this text.

The performance of HOPSO will be compared to that of the standard PSO, COBYLA [49], and Differential Evolution (DE) [50] optimization methods.

TABLE 2: Commonly used test functions for optimization methods

Name	Functional Form	Modality	Initial range	$F_{\min}$	Dimension
Ackley	$-a\exp\left(-b\sqrt{\frac{1}{d}\sum_{i=1}^{d}x_i^2}\right) - \exp\left(\frac{1}{d}\sum_{i=1}^{d}\cos(cx_i)\right) + a + \exp(1)$	Multimodal	[-32.76,32.76]	0	10
Beale	$(1.5 - x_1 + x_1x_2)^2 + (2.25 - x_1 + x_1x_2^2) + (2.625 - x_1 + x_1x_2^3)$	Multimodal	[-5,5]	0	2
Cross-in-Tray	$-0.0001[\lvert\sin(x_1)\sin(x_2) \exp(\lvert 100 - \frac{\sqrt{x_1^2+x_2^2}}{\pi}\rvert)\rvert + 1]^{0.1}$	Multimodal	[-10,10]	-2.06261	2
Drop-Wave	$-\frac{1+\cos(12\sqrt{x_1^2+x_2^2})}{0.5(x_1^2+x_2^2)+2}$	Multimodal	[-5.12,5.12]	-1	2
Goldstein-Price	$[1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \cdot [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)]$	Multimodal	[-2,2]	3	2
Griewank	$\frac{1}{4000}\sum_{i=1}^{d}x_i^2 - \prod_{i=1}^{d}\cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$	Multimodal	[-600,600]	0	10
Levy	$\sin^2(\pi w_1) + \sum_{i=1}^{d-1}(w_i - 1)^2[1 + 10\sin^2(\pi w_i + 1)] + (w_d - 1)^2[1 + \sin^2(2\pi w_d)]$	Multimodal	[-10,10]	0	10
Michalewicz	$-\sum_{i=1}^{d}\sin(x_i)\left[\sin\left(\frac{ix_i^2}{\pi}\right)\right]^{2m}$	Multimodal	[0,π]	-4.687	5
Rastrigin	$10d + \sum_{i=1}^{d}[x_i^2 - 10\cos(2\pi x_i)]$	Multimodal	[-5.12,5.12]	0	10
Rosenbrock	$\sum_{i=1}^{d-1}[100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2]$	Unimodal	[-5,10]	0	10
Schwefel	$\sum_{i=1}^{d}[-x_i\sin(\sqrt{\lvert x_i\rvert})]$	Multimodal	[-500,500]	0	10
Sphere	$\sum_{i=1}^{d}x_i^2$	Unimodal	[-10,10]	0	5

Constrained optimization by linear approximation (COBYLA) is a gradient-free simplex based optimization method. It was first introduced by Michael J.D. Powell in 1994 [49]. To describe the algorithm briefly, it operates by creating a simplex, a polytope of $n + 1$ vertices for an n-dimensional space, and using the values of the objective function at the vertices of this simplex it approximates the objective function along with linear contraints, solving linear programming problems within a trust region. The simplex and the trust region are adjusted iteratively until the convergence is obtained. For more details refer to Powell's original paper [49].

Differential Evolution (DE) is an another meta-heuristic algorithm like PSO. DE is an optimization technique that begins with a set of possible solutions and gradually refines them. It generates new solutions by mixing existing ones and keeps the better options in each iteration. This repeated process continues until it finds the best solution. DE was first given by Rainer Storn and Kenneth Price in 1997 [50]. To briefly describe DE, a population of candidate solutions is initialized randomly. For each candidate, a mutant vector is generated by adding the weighted difference between two randomly selected population vectors to a third vector. This mutant vector is then recombined with the target vector, and a selection process determines whether the new vector replaces the target vector based on a fitness evaluation. This process is repeated iteratively until a stopping criterion is met. It was shown recently [51] that DE, although not being as popular as PSO, outperforms PSO in many cases, hence it is a natural choice for an optimizer for comparison.

To compare optimization algorithms, it is necessary to introduce a total "budget" of function evaluations. This budget represents the number of times the algorithm is permitted to query the objective function to assess its performance. Each optimization method utilizes this budget in various different ways, depending on its internal parameters and search strategies. In principle, the same method can achieve different results for different values of tuning parameters, so we use the commonly-used "standard settings" which result in a typically-good performance for each optimization method. By setting a specific budget for each function—typically 10,000 or 1,000 function evaluations—it becomes possible to relate and adjust the set of tunable parameters of a given optimization method to this set budget, and to allow for a fair and reasonable comparison among all optimization methods.

Specifically, here in this study Scipy's optimization module [52] was used to implement COBYLA and DE. The budget for DE is calculated as the product of population size (equivalent to number of particles in PSO) and the maximum number of generations (i.e. iterations). The default population size for DE is taken as the dimension of the cost function. For COBYLA, the number of function evaluations equates to the maxiter parameter (i.e. maximum iterations allowed). Meanwhile PSO was run via our own implementation but using the standard parameter settings found in the literature mentioned earlier. It is important to note that since COBYLA and DE are Scipy optimizers, they run up to the max-iterations which is set by their own convergence criteria. However, the tolerance for the convergence was changed so that all of the budget is utilised efficiently.

In PSO, the product of the number of particles and the number of iterations define the function evaluation count. The PSO's aforementioned parameter-values are widely accepted based on empirical studies and provide reasonable results for many types of problems. However, due to the narrow range of acceptable values, it can be particularly challenging to adjust the PSO to fit a specific problem's landscape without significantly impacting its performance. As a result, PSO is less tunable and it's performance can sometimes be less flexible compared to other optimizers that allow for a broader range of parameter tuning. As such, we utilize its standard settings, which are (as mentioned previously) $\chi = 0.729$ and c_1, c_2 being each 2.05.

As in the PSO, the number of particles and iterations directly defines the budget in HOPSO. The HOPSO settings are designed to mirror the equivalent parameters in PSO, ensuring a fair playing field between the optimizers. Namely, the HOPSO settings are $c_1 = c_2 = \omega = 1$, $t_{ul} = 2\pi$ and $m = 2.05$. The randomness inherent in the PSO is similarly incorporated into HOPSO via time sampling, albeit HOPSO employs only a single random variable as opposed to PSO's use of two random variables. But in both cases, the randomness is applied independently to each particle and each dimension.

The crucial difference which offers the superior advantage of HOPSO above PSO is the tunability of the damping parameter λ. While the PSO has a damping factor as well, HOPSO's λ term, as mentioned earlier, serves as an easily controlled free parameter, influential on the energy control of the system and therefore its convergence as well. While this term acts like a free parameter, provided here is a general guideline (or starting point) for controlling HOPSO's damping, which will provide reasonable initial results but may be adjusted as desired by the user. But how does the user select a reasonable value? We intuitively reason that if there is a higher budget of function evaluations, there is a less need for damping in the system – since less damping allows for a broader search, and vice versa. Since we have established a finite budget of function evaluations to compare the optimization methods for each function, we provide a relationship for setting the lambda term based on the budget. It relates as follows

$$\lambda \equiv s\left(\frac{B}{N}\right)^{-1} \tag{21}$$

where B is the number of function evaluations (i.e. budget), N is the total number of particles, and s is a scaling factor, which we set to 10. It is possible that a better value of λ may be found than that presented in (21), a value that will be specially tailored to the cost-function's landscape and its properties. Nonetheless, we figure that this balance between the freedom for the user in the selection of the λ-value while providing a general "starting selection" that demonstrates strong results initially, is a strong reasoning for the optimal efficiency of this algorithm. The initially-strong results is clear from the results that we present in the next section, where we compare HOPSO to the other optimization methods having followed the criteria of equation (21). Further empirical and mathematical analysis is needed to investigate the dependencies of λ and its relation to the search space.

The results that are presented here in Figs. 4,5,6 use a fixed λ that is obtained from the specific function's chosen budget (refer to Table 3), 10 particles, and a scaling factor s set to equal 10 for all the functions. However, for the functions in which the HOPSO performed worse compared to its fellow optimizers, an analysis was done in varying lambda (via the s scaling factor). The results are presented in , from which one can conclude that fine-tuning this scaling factor may further improve the strong results presented in the other figures with s set to equal 10 .

The results presented compare the performance of HOPSO, PSO, COBYLA, and DE on a diverse set of test functions. These functions, selected for their varying properties, challenge the optimizers across different optimization landscapes and features. HOPSO consistently outperforms the other methods in complex multimodal functions, such as Cross-in-Tray and Goldstein-Price, demonstrating strong convergence to minima. For functions with many local minima, such as Ackley, HOPSO shows superior performance compared to PSO and DE. However, COBYLA and DE prove more competitive in the Rosenbrock function – a valley shaped unimodal function.

As can be inferred from Tab. 3, in the case of Cross in tray and Sphere, three or more optimizers perform similarly. In the case of the Sphere function, the mean of the all optimizers reaches below 10E-13, which we consider to be zero. In six cases, HOPSO outperforms the three other optimizers. In the rest of the four cases, DE performs the best. All these results of HOPSO are obtained without fine-tuning of the hyper parameters similarly as was the case with the other optimizers. It can be seen that sometimes, fine-tuning the hyper parameters of HOPSO can improve the results. Out of the four cases where DE performs the best, having fine-tuned the scaling parameter, s, of HOPSO resulted in it outperforming DE on the functions Michalewicz and Rastrigin. This is shown in the Fig. 7.

In virtually all the cases, HOPSO is at least as good as PSO. Thus, in any case when one may consider using PSO for their optimization purposes, HOPSO merits consideration over the PSO method – as HOPSO is a more powerful optimization tool.

V. CONCLUSION

In the paper we present a new algorithm – coined as 'HOPSO'– based on the popular population-based PSO optimizer – incorporating physics-inspired principles. It offers a strong advantage against the standard PSO in the fact that it prevents velocity explosions and has better tuning features, i.e. it offers more individual control over the iterative process without the risk of stagnation or divergence.

We have shown the power of the new procedure through its application onto a set of standard benchmark test optimization functions and compared to other non-gradient methods including COBYLA and DE. HOPSO outperformed the standard PSO and COBYLA in all cases, thus being able to completely replace PSO. In most cases, it has also outperformed the DE method, demonstrating HOPSO as a significantly competitive, powerful, and efficient optimization algorithm.

ACKNOWLEDGMENT

The authors would like to thank Martin Friak and his team for valuable discussions.

REFERENCES

[1] D.H. Wolpert and W.G. Macready. No free lunch theorems for optimization. *IEEE Transactions on Evolutionary Computation*, 1(1):67–82, April 1997.
[2] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.
[3] J. Kennedy and R. Eberhart. Particle swarm optimization. In *Proceedings of ICNN'95 - International Conference on Neural Networks*, volume 4, pages 1942–1948 vol.4, November 1995.
[4] John H. Holland. Genetic Algorithms. *Scientific American*, 267(1):66–73, 1992.
[5] Rainer Storn and Kenneth Price. Differential Evolution – A Simple and Efficient Heuristic for global Optimization over Continuous Spaces. *Journal of Global Optimization*, 11(4):341–359, December 1997.
[6] Xin Yao, Yong Liu, and Guangming Lin. Evolutionary programming made faster. *IEEE Transactions on Evolutionary Computation*, 3(2):82–102, July 1999.
[7] Nikolaus Hansen, Sibylle D. Müller, and Petros Koumoutsakos. Reducing the time complexity of the derandomized evolution strategy with covariance matrix adaptation (CMA-ES). *Evol. Comput.*, 11(1):1–18, March 2003.
[8] John R. Koza. Genetic programming as a means for programming computers by natural selection. *Statistics and Computing*, 4(2):87–112, June 1994.
[9] Haiping Ma, Dan Simon, Patrick Siarry, Zhile Yang, and Minrui Fei. Biogeography-Based Optimization: A 10-Year Review. *IEEE Transactions on Emerging Topics in Computational Intelligence*, 1(5):391–407, October 2017.
[10] Barry Webster and Philip J. Bernhard. A local search optimization algorithm based on natural principles of gravitation. In Hamid R. Arabnia, editor, *Proceedings of the International Conference on Information and Knowledge Engineering. IKE'03, June 23 - 26, 2003, Las Vegas, Nevada, USA, Volume 1*, pages 255–261. CSREA Press, 2003.
[11] Osman K. Erol and Ibrahim Eksin. A new optimization method: Big bang–big crunch. *Advances in Engineering Software*, 37(2):106–111, 2006.
[12] Esmat Rashedi, Hossein Nezamabadi-pour, and Saeid Saryazdi. Gsa: A gravitational search algorithm. *Information Sciences*, 179(13):2232–2248, 2009. Special Section on High Order Fuzzy Sets.
[13] Ali Kaveh and Siamak Talatahari. A novel heuristic optimization method: charged system search. *Acta mechanica*, 213(3):267–289, 2010.
[14] Richard Formato. Central force optimization: a new metaheuristic with applications in applied electromagnetics. *Progress in electromagnetics research*, 77:425–491, 2007.
[15] Bilal Alatas. Acroa: Artificial chemical reaction optimization algorithm for global optimization. *Expert Systems with Applications*, 38(10):13170–13180, 2011.
[16] Abdolreza Hatamlou. Black hole: A new heuristic optimization approach for data clustering. *Information Sciences*, 222:175–184, 2013. Including Special Section on New Trends in Ambient Intelligence and Bio-inspired Systems.
[17] A. Kaveh and M. Khayatazad. A new meta-heuristic method: Ray optimization. *Computers & Structures*, 112-113:283–294, 2012.

Function	Function evaluations	F_{min}	Mean			
			HOPSO	PSO	COBYLA	DE
Ackely	10000	0	**0.0115**	1.3824	19.410	5.6180
Beale	1000	0	**0.0363**	0.0635	1.0368	0.1174
Cross in tray	10000	-2.0626	**-2.0626**	**-2.0626**	-1.7594	**-2.0626**
Drop wave	10000	-1	**-0.9841**	-0.9790	-0.3781	-0.9460
Goldstein-Price	1000	3	**4.080**	5.4463	70.427	6.240
Griewank	10000	0	**0.1033**	0.1471	21.920	1.0461
Levy	10000	0	**0.1749**	2.0717	20.081	1.5805
Michealwicz	10000	-4.687	-4.4980	-4.0539	-3.0879	**-4.5187**
Rastrigin	10000	0	11.980	14.298	55.091	**11.770**
Rosenbrock	10000	0	5.3834	7776.2	12.612	**1.1960**
Schwefel	10000	0	1002.1	1083	1621.5	**686.57**
Sphere	1000	0	**0**	**0**	**0**	**0**

TABLE 3: Results of the optimization using different optimizers on different test functions.

[18] Haifeng Du, Xiaodong Wu, and Jian Zhuang. Small-world optimization algorithm for function optimization. In Licheng Jiao, Lipo Wang, Xinbo Gao, Jing Liu, and Feng Wu, editors, *Advances in Natural Computation*, pages 264–273, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.

[19] Hamed Shah-Hosseini. Principal components analysis by the galaxy-based search algorithm: a novel metaheuristic for continuous optimisation. *International Journal of Computational Science and Engineering*, 6(1-2):132–140, 2011.

[20] Fereydoun Farrahi Moghaddam, Reza Farrahi Moghaddam, and Mohamed Cheriet. Curved space optimization: a random search based on general relativity theory. *arXiv preprint arXiv:1208.2214*, 2012.

[21] Marco Dorigo and Thomas Stützle. *Ant Colony Optimization*. MIT Press, 2004.

[22] Dervis Karaboga and Bahriye Basturk. A powerful and efficient algorithm for numerical function optimization: artificial bee colony (ABC) algorithm. *Journal of Global Optimization*, 39(3):459–471, November 2007.

[23] Xin-She Yang. *A New Metaheuristic Bat-Inspired Algorithm*, pages 65–74. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[24] H.A. Abbass. Mbo: marriage in honey bees optimization-a haplometrosis polygynous swarming approach. In *Proceedings of the 2001 Congress on Evolutionary Computation (IEEE Cat. No.01TH8546)*, volume 1, pages 207–214 vol. 1, 2001.

[25] XL Li. A new intelligent optimization-artificial fish swarm algorithm. *Doctor thesis, Zhejiang University of Zhejiang, China*, 27, 2003.

[26] Roth Martin and Wicker Stephen. *Termite: A swarm intelligent routing algorithm for mobilewireless Ad-Hoc networks*, pages 155–184. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.

[27] Pedro C. Pinto, Thomas A. Runkler, and João M. C. Sousa. Wasp swarm algorithm for dynamic max-sat problems. In Bartlomiej Beliczynski, Andrzej Dzielinski, Marcin Iwanowski, and Bernardete Ribeiro, editors, *Adaptive and Natural Computing Algorithms*, pages 350–357, Berlin, Heidelberg, 2007. Springer Berlin Heidelberg.

[28] Antonio Mucherino and Onur Seref. Monkey search: a novel metaheuristic search for global optimization. *AIP Conference Proceedings*, 953(1):162–173, 11 2007.

[29] Xueyan Lu and Yongquan Zhou. A novel global convergence algorithm: Bee collecting pollen algorithm. In De-Shuang Huang, Donald C. Wunsch, Daniel S. Levine, and Kang-Hyun Jo, editors, *Advanced Intelligent Computing Theories and Applications. With Aspects of Artificial Intelligence*, pages 518–525, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.

[30] Xin-She Yang and Suash Deb. Cuckoo search via lévy flights. In *2009 World Congress on Nature & Biologically Inspired Computing (NaBIC)*, pages 210–214, 2009.

[31] Yang Shiqin, Jiang Jianjun, and Yan Guangxing. A dolphin partner optimization. In *2009 WRI Global Congress on Intelligent Systems*, volume 1, pages 124–128, 2009.

[32] Xin-She Yang. Firefly algorithm, stochastic test functions and design optimisation. *International journal of bio-inspired computation*, 2(2):78–84, 2010.

[33] Alireza Askarzadeh and Alireza Rezazadeh. A new heuristic optimization algorithm for modeling of proton exchange membrane fuel cell: bird mating optimizer. *International Journal of Energy Research*, 37(10):1196–1204, 2013.

[34] Amir Hossein Gandomi and Amir Hossein Alavi. Krill herd: A new bio-inspired optimization algorithm. *Communications in Nonlinear Science and Numerical Simulation*, 17(12):4831–4845, 2012.

[35] Wen-Tsao Pan. A new fruit fly optimization algorithm: Taking the financial distress model as an example. *Knowledge-Based Systems*, 26:69–74, 2012.

[36] Seyedali Mirjalili, Seyed Mohammad Mirjalili, and Andrew Lewis. Grey wolf optimizer. *Advances in engineering software*, 69:46–61, 2014.

[37] S.N. Sivanandam and S.N. Deepa. *Introduction to Particle Swarm Optimization and Ant Colony Optimization*, pages 403–424. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[38] Georgios Papazoglou and Pandelis Biskas. Review and comparison of genetic algorithm and particle swarm optimization in the optimal power flow problem. *Energies*, 16(3), 2023.

[39] Janmenjoy Nayak, H Swapnarekha, Bighnaraj Naik, Gaurav Dhiman, and S Vimal. 25 years of particle swarm optimization: Flourishing voyage of two decades. *Archives of Computational Methods in Engineering*, 30(3):1663–1725, 2023.

[40] Feng Wang, Heng Zhang, Kangshun Li, Zhiyi Lin, Jun Yang, and Xiao-Liang Shen. A hybrid particle swarm optimization algorithm using adaptive learning strategy. *Information Sciences*, 436-437:162–177, 2018.

[41] Radha Thangaraj, Millie Pant, Ajith Abraham, and Pascal Bouvry. Particle swarm optimization: Hybridization perspectives and experimental illustrations. *Applied Mathematics and Computation*, 217(12):5208–5226, 2011.

[42] Jinwei Qiao, Guangyuan Wang, Zhi Yang, Xiaochuan Luo, Jun Chen, Kan Li, and Pengbo Liu. A hybrid particle swarm optimization algorithm for solving engineering problem. *Scientific Reports*, 14(1):8357, 2024.

[43] Ivana Miháliková, Matej Pivoluska, Martin Plesch, Martin Friák, Daniel Nagaj, and Mojmír Šob. The cost of improving the precision of the variational quantum eigensolver for quantum chemistry. *Nanomaterials*, 12(2), 2022.
[44] Xavier Bonet-Monroig, Hao Wang, Diederick Vermetten, Bruno Senjean, Charles Moussa, Thomas Bäck, Vedran Dunjko, and Thomas E. O'Brien. Performance comparison of optimization methods on variational quantum algorithms. *Phys. Rev. A*, 107:032407, Mar 2023.
[45] James Kennedy Maurice Clerc. The particle swarm - explosion, stability and convergence in a multidimensional complex space. *Transactions on Evolutionary Computation*, pages 58–73, 2002.
[46] Daniel Bratton and James Kennedy. Defining a standard for particle swarm optimization. In *2007 IEEE Swarm Intelligence Symposium*, pages 120–127, 2007.
[47] Talha Ali Khan and Sai Ho Ling. A novel hybrid gravitational search particle swarm optimization algorithm. *Engineering Applications of Artificial Intelligence*, 102:104263, 2021.
[48] S. Surjanovic and D. Bingham. Virtual library of simulation experiments: Test functions and datasets. https://www.sfu.ca/~ssurjano/optimization.html, 2023. Accessed: 2024-06-18.
[49] Michael J.D. Powell. A direct search optimization method that models the objective and constraint functions by linear interpolation. *Advances in Optimization and Numerical Analysis*, pages 51–67, 1994.
[50] Rainer Storn and Kenneth Price. Differential evolution – a simple and efficient heuristic for global optimization over continuous spaces. *Journal of Global Optimization*, 11(4):341–359, 1997.
[51] Adam P Piotrowski, Jaroslaw J Napiorkowski, and Agnieszka E Piotrowska. Particle swarm optimization or differential evolution—a comparison. *Engineering Applications of Artificial Intelligence*, 121:106008, 2023.
[52] Pauli Virtanen, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, Stéfan J. van der Walt, Matthew Brett, Joshua Wilson, K. Jarrod Millman, Nikolay Mayorov, Andrew R. J. Nelson, Eric Jones, Robert Kern, Eric Larson, C J Carey, İlhan Polat, Yu Feng, Eric W. Moore, Jake VanderPlas, Denis Laxalde, Josef Perktold, Robert Cimrman, Ian Henriksen, E. A. Quintero, Charles R. Harris, Anne M. Archibald, Antônio H. Ribeiro, Fabian Pedregosa, Paul van Mulbregt, and SciPy 1.0 Contributors. SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. *Nature Methods*, 17:261–272, 2020.

YURY CHERNYAK received a B.A. degree in Mathematics and Physics at Hartwick College, and his M.S. degree in Physics from SUNY University at Albany in 2022. He began his doctoral studies at the Institute of Physics, Slovak Academy of Sciences in 2023.

IJAZ AHAMED MOHAMMAD is a senior Physics PhD student specializing in quantum computation. He finished his B.S and M.S in Physics at Indian Institute of Science Educational Research(IISER), Mohali, India in 2021. He was an INSPIRE-SHE scholarship recipient from 2016-2018. He started his doctoral studies at Institute of Physics, Slovak Academy of Sciences in 2021. He has contributed two research articles in the field of quantum information and computation.

NIKOLAS MASNICAK Recieved his B.S. and M.S. degree in theoretical physics from Masaryk University in Brno, Czech Republic in 2022. He started his doctoral studies at Institute of Physics, Slovak Academy of Sciences in 2023.

MATEJ PIVOLUSKA is a Senior Engineer at Quantum Technology Laboratories in Vienna, specializing in quantum optics, quantum cryptography, quantum information processing, and quantum computing. With a PhD in Informatics from Masaryk University, he has held various research positions, including at the Austrian Academy of Sciences and the Slovak Academy of Sciences. Pivoluska has contributed extensively to high-dimensional quantum key distribution, quantum entanglement, and quantum computing, with over 30 published articles.

MARTIN PLESCH is a physicist specializing in complex physical systems and quantum information theory. He is an independent researcher at the Institute of Physics Slovak Academy of Sciences and a Professor at Matej Bel University in Bratislava. With a PhD from the Institute of Physics Slovak Academy of Sciences, Prof. Plesch has held significant roles, including Head of the Department of Complex Physical Systems and Marie Curie Fellow at Masaryk University Brno. He has received numerous accolades for his research and educational contributions, including the Prize for Popularization of Science and the "Social Innovator" award. Prof. Plesch is also actively involved in international scientific committees and educational initiatives, serving as a President of the International Young Physicists' Tournament and the World Federation of Physics.

• • •

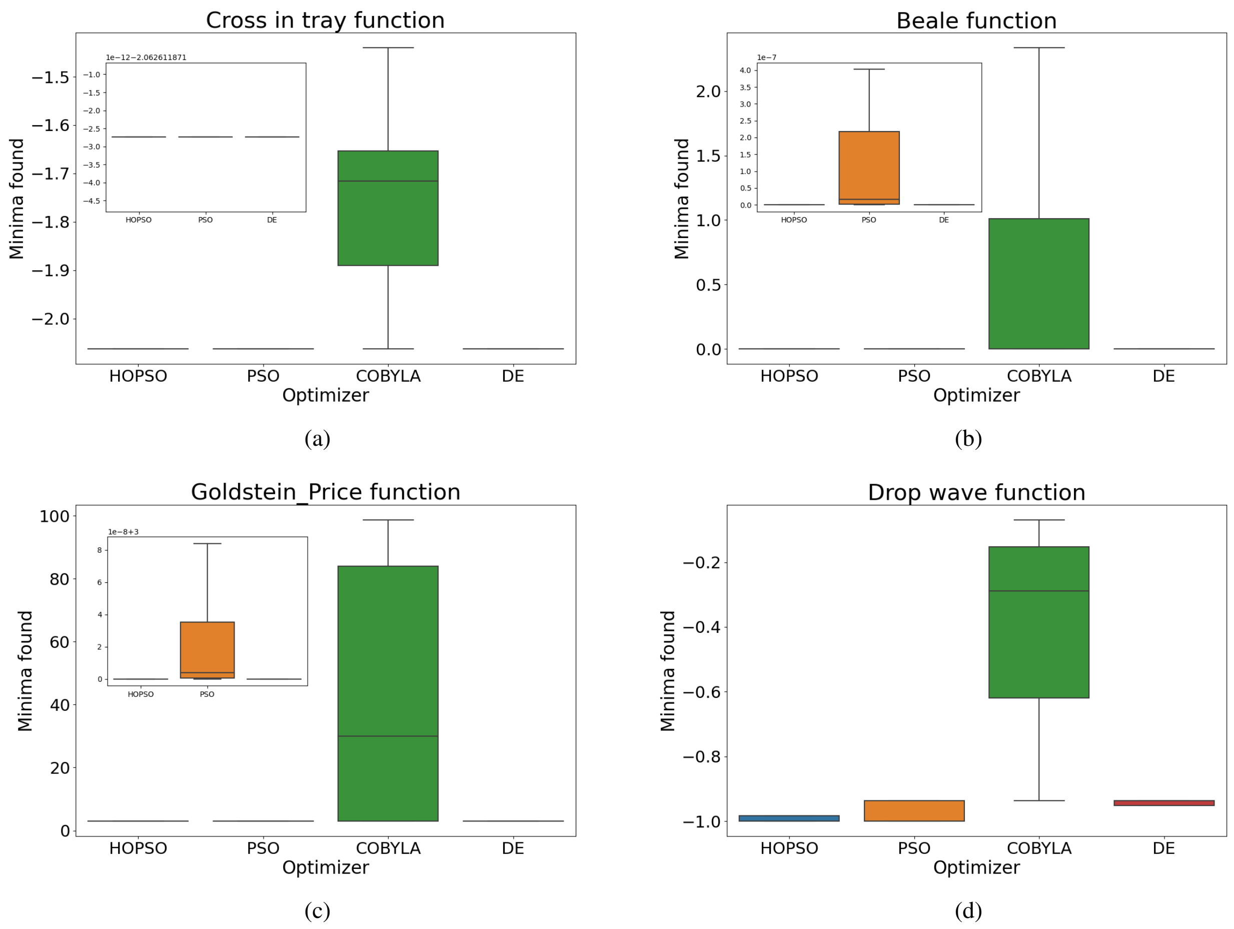

FIGURE 4: Comparison of the performance of different optimizers on constant dimensional ($dimension = 2$) test functions. (a) **Cross-in-Tray**: Apart from COBYLA, all the other optimizers converge to a minima, with HOPSO and PSO performing the best. (b) **Beale**: Except for COBYLA, all other optimizers converge to a minima, with HOPSO showing the greatest performance. (c) **Goldstein-Price**: Here COBYLA and PSO do not converge to the optimum unlike DE and HOPSO, with the latter outperforming its rival counterparts. (d) **Drop Wave**: All optimizers except HOPSO fail to converge to a global minima.

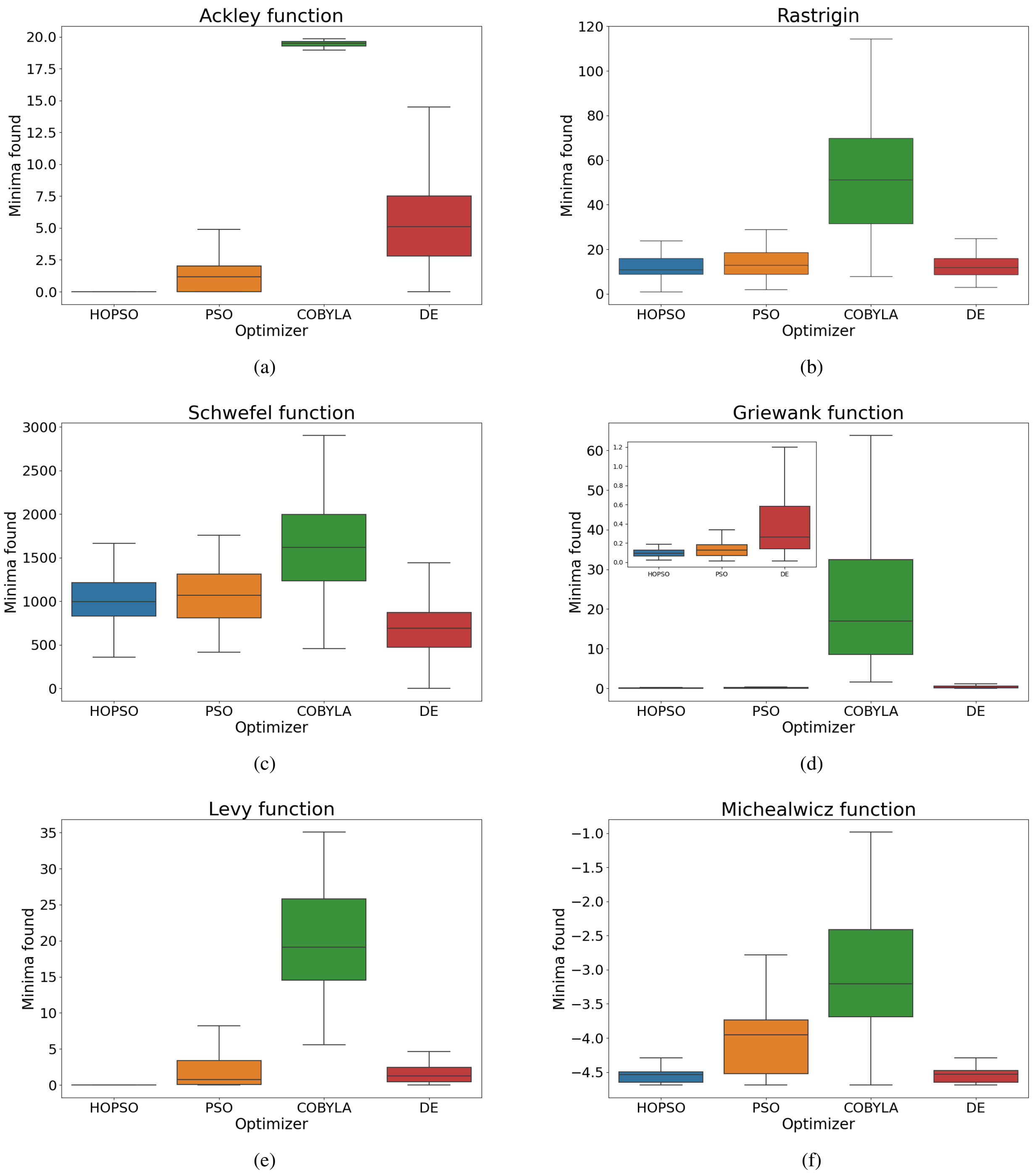

FIGURE 5: A comparison of the performance of different optimizers on varying-dimensional test functions. All test functions here are chosen to have a $\textit{dimension} = 10$ except for Michealwicz, whose *dimension* is chosen to be 5. (a) **Ackley**: All optimizers except HOPSO fail to converge to the minima. (b) **Rastrigin**: All optimizers fail to converge to the minima. However, HOPSO performs the best among them. (c) **Schwefel**: All optimizers fail to converge to the minima. Here however, DE performs the best. (d) **Griekwank**: All optimizers fail to converge to the minima. However, HOPSO's performance is significantly better compared to the other optimizers. (e) **Levy**: Except for HOPSO, all optimizers fail to converge to the minima. (f) **Michealwicz**: All optimizers fail to converge to the minima. HOPSO and DE perform the best among them.

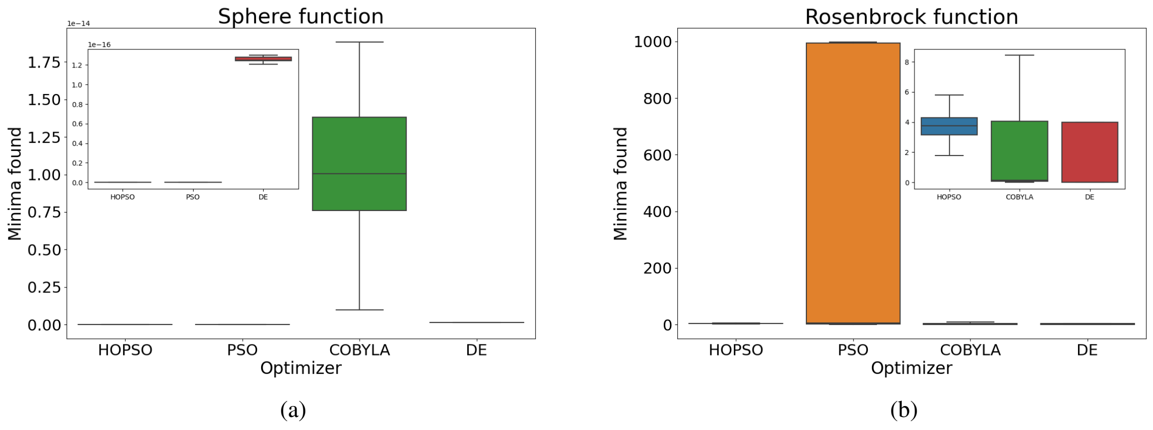

FIGURE 6: A comparison of the performance of four different optimizers on Unimodal test functions. (a) **Sphere**: All optimizers converge to the minima. For higher precision scenarios, HOPSO and PSO perform the best. (b) **Rosenbrock**: All optimizers fail to converge to the minima every time. However, DE and COBYLA perform better in comparison to PSO and HOPSO.

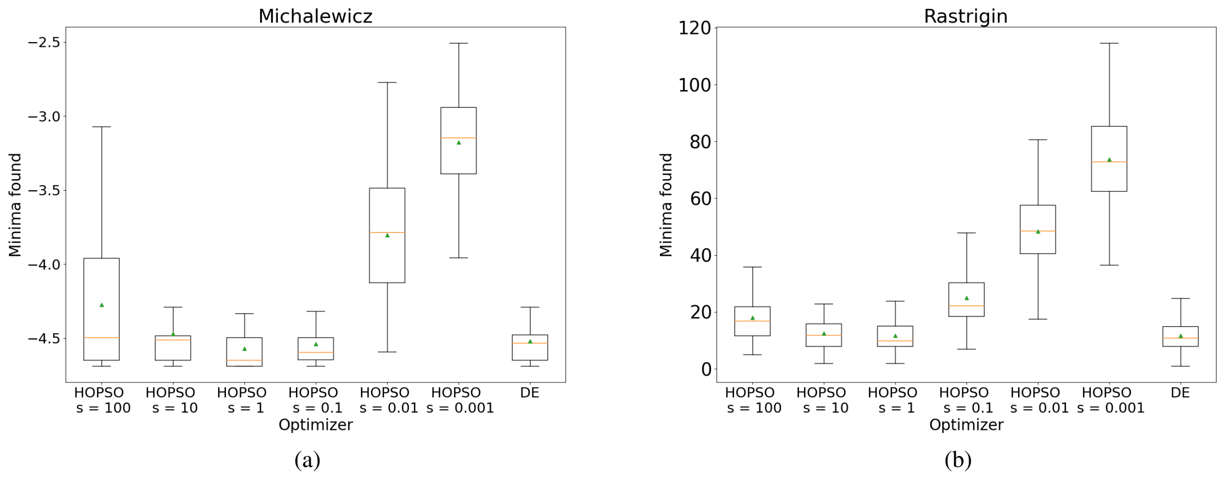

FIGURE 7: Comparison of the performance of HOPSO with varying λ based on the scaling factor s with DE on Michalewicz and Rastrigin functions. The green triangle represents the mean of the distribution and the yellow line representing the median. It can be seen that by fine-tuning the scaling factor HOPSO outperforms DE. (a) **Michalewicz** : The results of HOPSO with scaling factor set to 0.1 and 1 outperform the previously best-performing optimizer, DE. (b) **Rastrigin**: The results of HOPSO with the setting of the scaling factor s to 1 outperforms the previously best-performing optimizer, DE.